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The Problem of Relativity in reference to several bodies. By Mr R. HARGREAVES, St John's College.

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§ 1. If the kinetic potential for the relative motion of two masses is written with an added constant as

$$2L = \frac{m_1 m_2}{m_1 + m_2}\left\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - V^2 + \frac{2(m_1 + m_2)}{r}\right\},$$

a close connexion with the relativity quadratic appears. The latter is in fact

$$dx^2 + dy^2 + dz^2 + (\gamma^{-1} - 1)\, dr^2 - (V^2 - 2m/r)\, dt^2,$$

where $$\gamma = 1 - 2m/V^2 r;$$

a modification of the primary form

$$dx^2 + dy^2 + dz^2 - V^2 dt^2,$$

which shows an unaltered determinant. The condition in respect to the determinant, suggested, I believe, by Schwarzschild, is one which to me appears to give the most significant form to the results. From the dynamical standpoint we may regard it as imposing a counterpoise in the inertia coefficients to the modification introduced by the potential; or from a geometrical point of view we may regard it as minimizing the departure from the normal use of coordinates. An illuminating example of the loss of meaning that accompanies transformation in which this condition is disregarded is furnished by the isotropic form which is sometimes given to Einstein's quadratic.

It is proposed to examine whether it is practicable to apply a procedure suggested by the above considerations, when more than two bodies are concerned. The enquiry is prompted by a conviction that any attempt to modify $dx^2 + dy^2 + dz^2 - V^2 dt^2$ by use of a form based on accelerations to two or more *fixed* centres is inadmissible, on the ground that it fails to acknowledge the essential dynamical relativity of the bodies; or in other words the method is open to the objection that it does not deal with all bodies on equal terms, another way of stating the breach of the principle of relativity involved.

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It appears to me necessary to use a space quadratic* which shall include all relative coordinates, and one time coordinate. For this purpose one of the masses m is chosen arbitrarily as a centre from which rectangular coordinates are measured in fixed directions. Thus another mass m_p has coordinates x_p, y_p, z_p relative to m, and M may be used for the sum of masses $m + m_1 + m_2 + \ldots$. The primary form for several bodies is then taken to be

$$\left.\begin{aligned} & mm_1M^{-1}(dx_1^2 + dy_1^2 + dz_1^2 - V^2dt^2) \\ & \qquad + mm_2M^{-1}(dx_2^2 + dy_2^2 + dz_2^2 - V^2dt^2) + \ldots \\ & + m_1m_2M^{-1}\{(dx_2 - dx_1)^2 + (dy_2 - dy_1)^2 \\ & \qquad + (dz_2 - dz_1)^2 - V^2dt^2\} + \ldots \end{aligned}\right\} \quad (1).$$

As a first step terms

$$2dt^2(mm_1r_1^{-1} + mm_2r_2^{-1} + \ldots + m_1m_2r_{12}^{-1} + \ldots)$$

are added, which put us in touch with the kinetic potential for the relative motion of several bodies. It is then assumed that terms added to inertia take the shapes $\mu_1 dr_1^2/r_1$ or $\mu_{12}dr_{12}^2/r_{12}$ in the respective brackets of (1), and we seek to ascertain whether the values of the μ's admit of an adjustment which will make the inertial terms act as a counterpoise to potential terms, in the sense that the determinant of the modified quadratic agrees with that of the original. In using $\mu dr^2/r$ we have in mind the approximate value of $\gamma^{-1} - 1$ above, viz. $2m/rV^2$; our main concern is with a first approximation, though the exact problem will be briefly considered.

§ 2. The method will be applied to the case of four bodies m, m_1, m_2, m_3; the argument is quite general, but can be conducted with a less cumbrous notation than that required for n bodies. The coordinates $x_1, y_1, z_1 \ldots z_3$ are counted as having places 1, 2, 3 ... 9 in the scheme of space variables, and $a_{11} \ldots a_{99}$ are the elements of the space-determinant of the *unmodified* form (1). The property $a_{rs} = a_{sr}$ is general, and the special values of the a's for four bodies are:

$$\left.\begin{aligned} a_{11} = a_{22} = a_{33} &= m_1(M - m_1)/M = a \text{ say} \\ a_{44} = a_{55} = a_{66} &= m_2(M - m_2)/M = b \text{ say} \\ a_{77} = a_{88} = a_{99} &= m_3(M - m_3)/M = c \text{ say} \\ a_{14} = a_{25} = a_{36} &= -m_1m_2/M = h \text{ say} \\ a_{47} = a_{58} = a_{69} &= -m_2m_3/M = f \text{ say} \\ a_{71} = a_{82} = a_{93} &= -m_3m_1/M = g \text{ say} \end{aligned}\right\} \quad \ldots\ldots(2).$$

* The need for recognition of change in all space variables is emphasized by M. Le Roux, as I see in a quotation from *Comptes Rendus* given by Sir Joseph Larmor, who also presses the same point, *Proc. Camb. Phil. Soc.*, vol. XXI, pp. 415 *sqq.*

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With other suffixes the a's have a zero value, as a_{12} or a_{21}, since $dx\,dy$ does not occur. The determinant D of the space-quadratic is then

$$\begin{vmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 & a_{17} & 0 & 0 \\ 0 & a_{22} & 0 & 0 & a_{25} & 0 & 0 & a_{28} & 0 \\ 0 & 0 & a_{33} & 0 & 0 & a_{36} & 0 & 0 & a_{39} \\ a_{41} & 0 & 0 & a_{44} & 0 & 0 & a_{47} & 0 & 0 \\ 0 & a_{52} & 0 & 0 & a_{55} & 0 & 0 & a_{58} & 0 \\ 0 & 0 & a_{63} & 0 & 0 & a_{66} & 0 & 0 & a_{69} \\ a_{71} & 0 & 0 & a_{74} & 0 & 0 & a_{77} & 0 & 0 \\ 0 & a_{82} & 0 & 0 & a_{85} & 0 & 0 & a_{88} & 0 \\ 0 & 0 & a_{93} & 0 & 0 & a_{96} & 0 & 0 & a_{99} \end{vmatrix}$$

which is equal to the product

$$\begin{vmatrix} a_{11} & a_{14} & a_{17} \\ a_{41} & a_{44} & a_{47} \\ a_{71} & a_{74} & a_{77} \end{vmatrix} \times \begin{vmatrix} a_{22} & a_{25} & a_{28} \\ a_{52} & a_{55} & a_{58} \\ a_{82} & a_{85} & a_{88} \end{vmatrix} \times \begin{vmatrix} a_{33} & a_{36} & a_{39} \\ a_{63} & a_{66} & a_{69} \\ a_{93} & a_{96} & a_{99} \end{vmatrix} \quad (3).$$

The equality depends on the places of zero values in D, not on the special values shown in (2). The first minors of D are given by

$$\left.\begin{aligned} D_{11} &= \begin{vmatrix} a_{44} & a_{47} \\ a_{74} & a_{77} \end{vmatrix} \times \begin{vmatrix} a_{22} & a_{25} & a_{28} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \times \begin{vmatrix} a_{33} & a_{36} & a_{39} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \\ D_{41} &= -\begin{vmatrix} a_{14} & a_{17} \\ a_{74} & a_{77} \end{vmatrix} \times \begin{vmatrix} a_{22} & a_{25} & a_{28} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \times \begin{vmatrix} a_{33} & a_{36} & a_{39} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \\ D_{71} &= \begin{vmatrix} a_{14} & a_{17} \\ a_{44} & a_{47} \end{vmatrix} \times \begin{vmatrix} a_{22} & a_{25} & a_{28} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \times \begin{vmatrix} a_{33} & a_{36} & a_{39} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{vmatrix} \end{aligned}\right\} \quad (4).$$

For the values of D_{22}, D_{52}, D_{82} we take minors of the middle third-order determinant in (3) and associate them with the product of the other two. All first minors of D which correspond to zero places in D vanish, as $D_{21} = 0$.

§ 3. When account is taken of such terms as $\mu_1 dr_1^2/r_1$ in the modified form, the coefficients are more complicated, and none have zero value nor do any first minors vanish. But since a minor such as D_{21} is zero when μ's are absent, it follows that a term $a_{21}D_{21}$ is of the second order of small quantities. Therefore in the approximate evaluation of the determinant we may count places which had zero values as still zero, while taking account of modifications in the a's and in first minors for terms which had value in the original scheme.

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Thus to trace the effect of μ_1 we note that the addition to the quadratic is

$$mm_1 M^{-1}\mu_1 dr_1^2/r_1$$

or $$mm_1 M^{-1}\mu_1 (x_1 dx_1 + y_1 dy_1 + z_1 dz_1)^2/r_1^3,$$

and so with dashed letters for modified values

$$\left.\begin{aligned} a_{11}' &= a_{11} + mm_1 M^{-1}\mu_1 x_1^2/r_1^3 \\ a_{22}' &= a_{22} + mm_1 M^{-1}\mu_1 y_1^2/r_1^3 \\ a_{33}' &= a_{33} + mm_1 M^{-1}\mu_1 z_1^2/r_1^3 \end{aligned}\right\} \quad \ldots\ldots(5),$$

and it is not necessary to write a_{21}' .... The first of the product determinants in (3) is replaced by

$$\begin{vmatrix} a + mm_1 M^{-1}\mu_1 x_1^2/r_1^3 & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

which is $$\Delta + mm_1 M^{-1}\mu_1 x_1^2 r_1^{-3} (bc - f^2) \quad \ldots\ldots(6),$$

where Δ is the original form of each of the third order determinants. With the values in (2)

$$\Delta = mm_1 m_2 m_3/M, \quad bc - f^2 = m_2 m_3 (m + m_1)/M,$$

and therefore (6) is $\Delta \left(1 + \dfrac{m + m_1}{M} \cdot \dfrac{\mu_1 x_1^2}{r_1^3}\right)$.

The second and third of the determinants in (2) differ only in having y_1^2 and z_1^2 in lieu of x_1^2, and accordingly the value of D as modified by μ_1 only is

$$\Delta^3 \left(1 + \frac{m + m_1}{M} \cdot \frac{\mu_1}{r_1}\right).$$

There is complete symmetry in the quadratic form, and therefore, so long as we attend only to first order corrections, the whole modified determinant is

$$\Delta^3 \left(1 + \frac{m + m_1}{M}\frac{\mu_1}{r_1} + \ldots + \frac{m_1 + m_2}{M}\frac{\mu_{12}}{r_{12}} + \ldots\right) \ldots\ldots(7).$$

A little more algebra is needed to reach the final form in the case of μ_{12}, but in view of the manifest symmetry it seemed sufficient to take the simplest case.

If now we write P for the sum of products of masses two and two, $mm_1 + \ldots + m_1 m_2 + \ldots$, the increment in the determinant of *tenth* order where time is included, due to the change in the spatial section, is

$$-\frac{PV^2}{M} \times \Delta^3 \left(\frac{m + m_1}{M}\frac{\mu_1}{r_1} + \ldots\right),$$

while the increment due to the potential terms which modify the coefficient of dt^2 is

$$2\Delta^3 \left(\frac{mm_1}{r_1} + \ldots\right).$$

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When these are made to balance we have

$$\mu_1 V^2 = 2mm_1 M^2/(m + m_1)\, P, \quad \mu_{12} V^2 = 2m_1 m_2 M^2/(m_1 + m_2)\, P, \ldots \quad \ldots\ldots(8).$$

Thus with a choice of values for the μ's dependent solely on the masses concerned, the modified determinant of the tenth order retains the value of the original. It will be observed that in the formula for μ_{12} other masses than m_1 and m_2 enter only through the values of M and P; if only m and m_1 exist then μ_1 becomes $2(m + m_1)/V^2 r$, the primary case from which we set out. Also it may be well to state, in respect to the problem for any number of bodies, that the determinant of the unmodified space quadratic is always the cube of a constituent Δ, the order of which is one less than the number of bodies. For three bodies it has a form

$$\begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}$$

equal in value to $\begin{vmatrix} a_{22} & a_{25} \\ a_{52} & a_{55} \end{vmatrix}$ or $\begin{vmatrix} a_{33} & a_{36} \\ a_{63} & a_{66} \end{vmatrix}$,

while for five bodies Δ has a first row

$$a_{11} \quad a_{14} \quad a_{17} \quad a_{1,10}.$$

§ 4. The question is now what treatment is to be applied to the modified quadratic we have reached, as for example to that of the seventh order required for the relative motion of three bodies. As I understand the matter, the primary treatment is to take

$$dx_1/ds, \ldots d(Vt)/ds$$

as velocities, and apply the usual variation method to a kinetic potential

$$L = \frac{mm_1}{2M}\left\{\left(\frac{dx_1}{ds}\right)^2 + \ldots\right\} + \ldots$$

written with all the modifying terms. This is a purely kinetic system with seven velocities, and t is an ignorable coordinate.

I have given elsewhere* a method which combines the ignoration with a transfer to the use of the ignored coordinate as independent variable. The application of this method yields a new kinetic potential

$$L = \frac{mm_1}{2M\gamma}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 + \mu_1 \dot{r}_1^2/r_1) + \ldots - \frac{PV^2}{2M} + \frac{mm_1}{r_1} + \frac{mm_2}{r_2} + \frac{m_1 m_2}{r_{12}} \quad \ldots\ldots(9),$$

* *Proc. Camb. Phil. Soc.*, vol. XXII, Part 2, § 17. There is a divisor h^2 affecting the gravitation constant, where h is the constant value of $\gamma d(Vt)/ds$; in the above this is taken to be 1, or to be absorbed in the gravitation constant.

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$\dot{x}$ being dx/dt and

$$\gamma = 1 - \frac{2M}{PV^2}\left(\frac{mm_1}{r_1} + \frac{mm_2}{r_2} + \frac{m_1 m_2}{r_{12}}\right).$$

Or we may say that the usual formula

$$\frac{mm_1}{2M}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \ldots + \frac{mm_1}{r_1} + \frac{mm_2}{r_2} + \frac{m_1 m_2}{r_{12}}$$

is to be supplemented by first-order modifications

$$\frac{1}{2M}(mm_1\mu_1\dot{r}_1^2/r_1 + mm_2\mu_2\dot{r}_2^2/r_2 + m_1 m_2 \dot{r}_{12}^2/r_{12})$$
$$+ \frac{1}{PV^2}\{mm_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \ldots\}\left(\frac{mm_1}{r_1} + \frac{mm_2}{r_2} + \frac{m_1 m_2}{r_{12}}\right) \quad (10).$$

Thus the method, as here carried through, leads to a definite modification of the normal kinetic potential belonging to the problem of n bodies; and, as noted at the end of § 3, the supplementary terms are, in the case of two bodies only, reduced to the corrections given by Einstein.

A brief summary of the argument may be added. From the dynamical point of view it is a striking feature of Einstein's quadratic form that it provides by ignoration of t the potential terms of dynamics. It is assumed that this is also the case with several bodies, so that the content of a modified coefficient of dt^2 is dictated by our knowledge of the potential. By analogy with the case of two bodies compensatory terms varying as dr^2 are introduced, and if the type is admitted the coefficients follow from the principle that the determinant is unchanged in value.

§ 5. Curiosity led me to examine, in the case of three bodies, the exact value of the spatial determinant. For that purpose the quadratic may be written as

$$A(dx_1^2 + dy_1^2 + dz_1^2) + B(dx_2^2 + dy_2^2 + dz_2^2)$$
$$+ C\{(dx_2 - dx_1)^2 + \ldots\} + a\,dr_1^2 + b\,dr_2^2 + c\,dr_{12}^2 \quad \ldots(11).$$

With $dr_1 = l_1 dx_1 + m_1 dy_1 + n_1 dz_1$, $\quad dr_{12} = l_3(dx_2 - dx_1) + \ldots$

we have three systems of direction-cosines connected by

$$l_3 r_{12} = x_2 - x_1 = l_2 r_2 - l_1 r_1, \ldots.$$

The determinant is then

$$\begin{array}{|cccccc|}
A + C + al_1^2 + cl_3^2, & al_1m_1 + cl_3m_3, & al_1n_1 + cl_3n_3, & -C - cl_3^2, & -cl_3m_3, & -cl_3n_3 \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
-C - cl_3^2, & -cl_3m_3, & -cl_3n_3, & B + C + bl_2^2 + cl_3^2, & bl_2m_2 + cl_3m_3, & bl_2n_2 + cl_3n_3 \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots
\end{array} \quad \ldots\ldots(12).$$

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The part independent of a, b, c is $(AB + BC + CA)^3$ or Δ^3. Each of the letters a, b, c may occur in the first but in no higher power; the value of the determinant is

$$\left.\begin{aligned}&\Delta^3 + \Delta^2\{a(B+C) + b(C+A) + c(A+B)\} + \Delta^2(ab+bc+ca)\\&\quad + 4\Delta S^2(abC^2r_1^{-2}r_2^{-2} + bcA^2r_2^{-2}r_{12}^{-2} + caB^2r_1^{-2}r_{12}^{-2})\\&\quad + 4\Delta S^2abc(Cr_1^{-2}r_2^{-2} + Ar_2^{-2}r_{12}^{-2} + Br_1^{-2}r_{12}^{-2}).\end{aligned}\right\} \quad (13).$$

The direction-cosines occur in combinations which may be expressed in terms of the mutual distances. There is, for example, a term

$$ab\Delta\{(A+C)(B+C) - C^2(\Sigma l_1 l_2)^2\}$$

which is

$$ab\Delta(\Delta + 4C^2S^2r_1^{-2}r_2^{-2}),$$

where S is the area of the triangle formed by the three point-masses.

We may alter the notation to bring it into closer relation with previous work by setting $a = A\lambda_1$, $b = B\lambda_2$, $c = C\lambda_{12}$, so that λ_1 takes the place of μ_1/r_1. Since A, B, C, Δ have the values mm_1M^{-1}, mm_2M^{-1}, $m_1m_2M^{-1}$, $mm_1m_2M^{-1}$ the value of the determinant (12) is

$$\left.\begin{aligned}&\Delta^3[1 + M^{-1}\{(m+m_1)\lambda_1 + (m+m_2)\lambda_2 + (m_1+m_2)\lambda_{12}\}\\&\quad + M^{-1}(m\lambda_1\lambda_2 + m_1\lambda_1\lambda_{12} + m_2\lambda_2\lambda_{12})\\&\quad + 4S^2M^{-2}(m_1m_2\lambda_1\lambda_2r_1^{-2}r_2^{-2} + mm_1\lambda_2\lambda_{12}r_2^{-2}r_{12}^{-2}\\&\quad + mm_2\lambda_1\lambda_{12}r_1^{-2}r_{12}^{-2}) + 4S^2M^{-2}\lambda_1\lambda_2\lambda_3(m_1m_2r_1^{-2}r_2^{-2}\\&\quad + mm_1r_2^{-2}r_{12}^{-2} + mm_2r_1^{-2}r_{12}^{-2})]\end{aligned}\right\} \quad (14).$$

The property of the determinant that it depends solely on the mutual distances is interesting, and may probably be true for any number of bodies. I have no speculation to suggest in reference to terms beyond the first order; in the inertia section they can hardly be within the limits of observation, in the potential section they might be.

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