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On Cover Graphs and Dependent Arcs in Acyclic Orientations

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The present paper deals with two graph parameters related to cover graphs and acyclic orientations of graphs.

The parameter $c(G)$ of a graph G, introduced by B. Bollobás, G. Brightwell and J. Nešetřil [*Order* **3** 245–255], is defined as the minimum number of edges one needs to delete from G in order to obtain a cover graph. Extending their results, we prove that, for $\delta > 0$, $(1-\delta)\frac{1}{l}\frac{n^2p}{2} \leqslant c(\mathscr{G}_{n,p}) \leqslant (1+\delta)\frac{1}{l}\frac{n^2p}{2}$ asymptotically almost surely as long as $Cn^{-1+\frac{1}{l}} \leqslant p(n) \leqslant cn^{-1+\frac{1}{l-1}}$ for some positive constants c and C. Here, as usual, $\mathscr{G}_{n,p}$ is the random graph.

Given an acyclic orientation of a graph G, an arc is called dependent if its reversal creates an oriented cycle. Let $d_{\min}(G)$ be the minimum number of dependent arcs in any acyclic orientation of G. We determine the supremum, denoted by $r_{\chi,g}$, of $d_{\min}(G)/e(G)$ in the class of graphs G with chromatic number χ and girth g. Namely, we show that $r_{\chi,g} = \binom{\chi-\frac{g}{2}+2}{2}/\binom{\chi}{2}$. This extends results of D. C. Fisher, K. Fraughnaugh, L. Langley and D. B. West [*J. Combin. Theory Ser. B* **71** 73–78].

1. Introduction and results

Let $G = (V, E)$ be a graph. We let $v(G)$ denote the number of vertices of G and $e(G)$ denote the number of edges of G. A subgraph of G induced on a set of vertices $A \subseteq V(G)$ is denoted by $G[A]$. An *orientation* of G, denoted by $\vec{G}$, is a subset of $V \times V$ satisfying $|\vec{G} \cap \{(i,j),(j,i)\}| = 1$ for any edge $\{i,j\}$ of G and $|\vec{G} \cap \{(i,j),(j,i)\}| = 0$ otherwise. An orientation which does not contain an oriented cycle is called *acyclic*. Note that every graph G possesses at least one acyclic orientation.

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Definition 1. Let $P = (V, <_P)$ be a finite poset with an underlying set V. We define its cover graph $G = (V, E)$ by $\{x, y\}, x, y \in V$, being an edge if and only if $x <_P y$ and there is no $z \in P$ such that $x <_P z <_P y$.

Given a graph G one can ask how to measure how far G is from being a cover graph. The present paper deals with two graph parameters related to this question, $c(G)$ and $d_{\min}(G)$, defined below.

The parameter $c(G)$ is defined to be the minimum number of edges of a graph G which needs to be deleted from G to obtain a cover graph, *i.e.*,

$$c(G) = \min\{|F| : F \subseteq E(G), G \setminus F \text{ is a cover graph}\}. \tag{1.1}$$

The first part of our paper complements a result of Bollobás, Brightwell and Nešetřil in [4] who introduced and investigated $c(G)$.

Before defining $d_{\min}(G)$ we need to introduce the notion of a dependent arc.

Definition 2. Given an acyclic orientation of a graph, an arc is called *dependent* if the reversal of its orientation creates an oriented cycle.

For an acyclic orientation $\vec{G}$ of a graph G, we let $d(\vec{G})$ denote the number of dependent arcs in $\vec{G}$. We define $d_{\min}(G)$ to be the minimum number of dependent arcs in any acyclic orientation of a graph G, *i.e.*,

$$d_{\min}(G) = \min_{\vec{G}}(\text{the number of dependent arcs in } \vec{G}) = \min_{\vec{G}} d(\vec{G}),$$

where $\vec{G}$ runs over all acyclic orientations of G. We define $d_{\max}(G)$ similarly. The parameter $d_{\min}(G)$ was investigated by Fisher, Fraughnaugh, Langley and West [6]. Sharpening some of the results from [6], we determine asymptotically the maximum value of $d_{\min}(G)$ for graphs with fixed chromatic number and girth.

Note that $c(G) \leqslant d_{\min}(G)$ for every graph G. Moreover, the following obvious fact relates cover graphs, dependent arcs, and $d_{\min}(G)$.

Fact 1.1. *Let G be a graph. The following are equivalent:*

(1) *G is a cover graph (i.e., $c(G) = 0$),*
(2) *there is an acyclic orientation of G which possesses no dependent arcs,*
(3) *$d_{\min}(G) = 0$.*

1.1. Cover graphs via deleting edges

Our main result concerning $c(G)$, for G being the binomial random graph, is Theorem 1.2 given below.

As pointed out in [2], it follows from the construction of Descartes [5] that for every χ there exists a partial order with a cover graph G such that $\chi(G) \geqslant \chi$. It was shown by Bollobás [2] that this remains true even for G being a cover graph of a finite lattice. On the other hand, Nešetřil and Rödl [13], in response to a conjecture raised in [2], proved that for every $g \geqslant 3$ there are graphs G with girth$(G) \geqslant g$ such that G is not a cover graph.

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Let $\mathscr{G}_{n,p}$ be the probability space over all graphs on n vertices endowed with a product measure given by taking each edge independently with probability p. The objective of [4] was to investigate how the random variable $c(\mathscr{G}_{n,p})$ depends on p. For an integer $l \geqslant 2$, a real number $\delta > 0$, and probability $p(n) = o(n^{-1+1/(l-1)})$, Bollobás, Brightwell and Nešetřil [4] proved that

$$c(\mathscr{G}_{n,p}) \leqslant (1+\delta)\frac{1}{l} \cdot \frac{pn^2}{2}$$

holds a.a.s.[1] Further, they showed that, for $p(n) = n^{-1+\eta(n)}$ with $0 < \eta_0 \leqslant \eta(n) \leqslant 1$,

$$k_0 pn^2 \leqslant c(\mathscr{G}_{n,p})$$

a.a.s. for some $k_0 = k_0(\eta_0) > 0$. While the method used in [4] does not seem to allow computing the value of k_0, the authors expressed belief that, for $p(n)$ such that $\lim_{n\to\infty} p(n)n^{1-1/(l-1)} = 0$ and $\lim_{n\to\infty} p(n)n^{1-1/l} = \infty$, their upper bound is correct, *i.e.*, $c(\mathscr{G}_{n,p}) \sim \frac{pn^2}{2l}$ holds. In the following theorem we confirm this conjecture in a slightly stronger form.

Theorem 1.2. *Let an integer $l \geqslant 2$ and a real number $\delta, 0 < \delta < 1$, be given. There exist constants $c > 0$ and $C > 0$ such that, if*

$$Cn^{-1+\frac{1}{l}} \leqslant p(n) \leqslant cn^{-1+\frac{1}{l-1}}, \tag{1.2}$$

then

$$(1-\delta)\frac{1}{l} \cdot \frac{n^2 p}{2} \leqslant c(\mathscr{G}_{n,p}) \leqslant (1+\delta)\frac{1}{l} \cdot \frac{n^2 p}{2} \tag{1.3}$$

holds a.a.s.

1.2. Dependent arcs in acyclic orientations

This subsection introduces results on the asymptotic behaviour of the maximum value of $d_{\min}(G)$ for graphs with fixed chromatic number and girth.

Let us start with an example. Since K_χ possesses up to isomorphism only one acyclic orientation, it is straightforward to observe that

$$d_{\min}(K_\chi) = d_{\max}(K_\chi) = \binom{\chi-1}{2}. \tag{1.4}$$

Let $\chi \geqslant 2$ and $g \geqslant 3$ be integers. To describe the behaviour of $d_{\min}(G)$, we consider the following parameter introduced by Fisher, Fraughnaugh, Langley and West in [6]:

$$r_{\chi,g} = \sup_G \left\{ \frac{d_{\min}(G)}{e(G)} : \chi(G) = \chi,\ \operatorname{girth}(G) = g \right\}. \tag{1.5}$$

Note that $r_{2,g} = 0$ and also, for $g \geqslant 4$, $r_{3,g} = 0$. Considering the complete graph on three vertices K_3 one gets that $r_{3,3} = 1/3$. The authors of [6] showed $\frac{1}{20} \leqslant r_{4,4} \leqslant \frac{1}{2}$. They also proved the following more general upper bound.

[1] A sequence of events $\mathscr{E}_n$ occurs *asymptotically almost surely*, a.a.s., if $\operatorname{Prob}(\mathscr{E}_n) = 1 - o(1)$ as $n \to \infty$.

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Theorem 1.3 ([6]). *For integers $\chi \geqslant 2$ and $g \geqslant 3$,*

$$r_{\chi,g} \leqslant \frac{\chi - 2}{\chi}. \tag{1.6}$$

In our main result concerning $r_{\chi,g}$ we determine its exact values.

Theorem 1.4. *Let $\chi \geqslant 2$ and $g \geqslant 3$ be integers. Then*

$$r_{\chi,g} = \frac{\binom{\chi-g+2}{2}}{\binom{\chi}{2}}. \tag{1.7}$$

Note that (1.7) implies $r_{\chi,g} = 0$ if $\chi < g$, which is well known.

Finally, let us note two related results concerning the possible number of dependent arcs in acyclic orientations of a given graph G. Let $D(G)$ be the set of integers d for which there is an acyclic orientation of the graph G with precisely d dependent arcs. Fisher, Fraughnaugh, Langley and West [6] proved that if G is a graph with $\chi(G) < \text{girth}(G)$ then the set $D(G)$ is the interval $[0, d_{\max}(G)]$. Unlike $d_{\min}(G)$, the quantity $d_{\max}(G)$ does not depend on χ and g since Edelman showed that $d_{\max}(G) = e(G) - v(G) + \text{com}(G)$ (*cf.* [6]). Here $\text{com}(G)$ denotes the number of components of the graph G.

1.3. Structure of the paper

The paper is divided into 8 sections. The proof of Theorem 1.2 is presented in Section 2.

Sections 3 to 8 are devoted to the proof of Theorem 1.4. Sections 3 and 4 contain a proof of the upper and lower bounds on $d_{\min}(G)$. Whereas proving the upper bound is simple, the lower bound is based on a probabilistic construction of a sequence of graphs for which the value of $d_{\min}(G)$ asymptotically matches the upper bound. This construction is summarized in the technical Proposition 5.1 and its proof is given in Sections 5 to 8.

2. Proof of Theorem 1.2

Theorem 1.2 will follow from Lemmas 2.1 and 2.2. Let $\mathcal{G}(n, M)$ be the probability space over all graphs on n vertices and M edges endowed with the uniform measure.

Lemma 2.1. *Let an integer $l \geqslant 2$ and a real number $\delta, 0 < \delta < 1$, be given. Then for any M and M', $0 < M' \leqslant \frac{\delta}{2} M$, the following inequality holds:*

$$\begin{aligned} &\text{Prob}\left(\left\{G \in \mathcal{G}(n, M) : c(G) < (1-\delta)\frac{1}{l}M\right\}\right) \\ &\quad \leqslant \frac{M}{M'} \cdot \text{Prob}\left(\left\{G \in \mathcal{G}(n, M') : c(G) < \left(1 - \frac{\delta}{2}\right)\frac{1}{l}M'\right\}\right). \end{aligned} \tag{2.1}$$

Proof. Let $l \geqslant 2, \delta > 0, M$, and $M', 0 < M' \leqslant \frac{\delta}{2} M$, be given. Let $k = \lfloor M/M' \rfloor$. This choice implies $k \geqslant \frac{1}{\delta}$.

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We consider $\mathscr{G}(n,M)$ equivalently as the following random graph process $\mathscr{G}_m = ([n], E_m)$, $m = 0,1,\ldots,M$, where $E_0 = \emptyset$ and $\mathscr{G}_m$ is obtained from $\mathscr{G}_{m-1}$ by choosing an edge e_m uniformly at random from $\bar{E}_{m-1} = \binom{[n]}{2} \setminus E_{m-1}$ and putting $E_m = E_{m-1} \cup \{e_m\}$. We let $G_i = ([n], E_{iM'} \setminus E_{(i-1)M'})$, $i = 1,\ldots,k$, and $G_0 = ([n], E_M \setminus E_{kM'})$. Note that $|E(G_i)| = M'$, $i = 1,\ldots,k$, and $|E(G_0)| < M' \leqslant \frac{\delta}{2}M$. Our choice of M' implies that if $c(G) < (1-\delta)\frac{1}{l}M$, then $c(G) < (1-\frac{\delta}{2})\frac{1}{l}M'k$. By averaging, there exists an $i \in \{1,\ldots,k\}$ such that $c(G_i) < (1-\frac{\delta}{2})\frac{1}{l}M'$. Since each G_i, $i = 1,\ldots,k$, is equally likely to be any graph on n vertices containing M' edges, we infer

$$\begin{aligned}
&\mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M) : c(G) < (1-\delta)\frac{1}{l}M\right\}\right) \\
&\quad \leqslant k \cdot \mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M') : c(G) < \left(1-\frac{\delta}{2}\right)\frac{1}{l}M'\right\}\right) \\
&\quad \leqslant \frac{M}{M'} \cdot \mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M') : c(G) < \left(1-\frac{\delta}{2}\right)\frac{1}{l}M'\right\}\right). \qquad \square
\end{aligned}$$

Lemma 2.2. *Let an integer $l \geqslant 2$ and a real number $\delta, 0 < \delta < 1$, be given. There exist positive real numbers $c_1 = c_1(l,\delta)$ and $c_2 = c_2(l,\delta) > c_1$ such that, for any M satisfying $c_1 n^{1+1/l} \leqslant M \leqslant c_2 n^{1+1/l}$, the following holds:*

$$\mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M) : c(G) < (1-\delta)\frac{1}{l}\cdot M\right\}\right) = o(n^{-1}). \tag{2.2}$$

We will postpone the proof of Lemma 2.2 to the end of the section. First we will show how it implies Theorem 1.2.

Proof of Theorem 1.2. Let $l \geqslant 2$ and $\delta > 0$ be given. The proof of the upper bound in (1.3), and thus the existence of a constant $c > 0$ (*cf.* (1.2)), follows the proof in [4] and therefore we omit it here. (Note that the upper bound in (1.3) was claimed in [4] under a slightly stronger assumption $\lim_{n\to\infty} p(n)n^{1-\frac{1}{l-1}} = 0$.)

We turn to proving the lower bound in (1.3). Let $c_1 = c_1(l,\delta/4)$ and $c_2 = c_2(l,\delta/4)$ be the real numbers guaranteed by Lemma 2.2. Set $C = \frac{16}{\delta}c_2$.

We claim that for every M, $\frac{1}{4}Cn^{1+\frac{1}{l}} \leqslant M \leqslant cn^{1+\frac{1}{l-1}}$, we have

$$\mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M) : c(G) < \left(1-\frac{\delta}{2}\right)\frac{1}{l}M\right\}\right) = o(1). \tag{2.3}$$

Indeed, let $M' = \lceil c_1 n^{1+1/l}\rceil$. Since M' satisfies $c_1 n^{1+1/l} \leqslant M' \leqslant c_2 n^{1+1/l}$, Lemma 2.2 implies

$$\mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M') : c(G) < \left(1-\frac{\delta}{4}\right)\frac{1}{l}M'\right\}\right) = o(n^{-1}).$$

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Since our choice of C implies $M' \leqslant \frac{1}{2} \cdot \frac{\delta}{2} M$, we infer, using Lemma 2.1,

$$\begin{aligned}
&\mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M) : c(G) < \left(1-\frac{\delta}{2}\right)\frac{1}{l}M\right\}\right) \\
&\quad \leqslant \frac{M}{M'} \cdot \mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M') : c(G) < \left(1-\frac{\delta}{4}\right)\frac{1}{l}M'\right\}\right) \\
&\quad = \frac{cn^{1+\frac{1}{l-1}}}{c_1 n^{1+\frac{1}{l}}} \cdot o(n^{-1}) \\
&\quad = o(1),
\end{aligned}$$

which is claimed in (2.3). We are going to show that our claim in (2.3) already implies the lower bound in (1.3). Set

$$P_1(M) = \mathrm{Prob}\left(\left\{G \in \mathscr{G}_{n,p} : c(G) < (1-\delta)\frac{1}{l} \cdot \frac{n^2 p}{2} \mid e(G) = M\right\}\right).$$

Indeed, we have

$$\mathrm{Prob}\left(\left\{G \in \mathscr{G}_{n,p} : c(G) < (1-\delta)\frac{1}{l} \cdot \frac{n^2 p}{2}\right\}\right) \leqslant P_1 + P_2,$$

where

$$P_1 = \sum_{|M - \frac{n^2 p}{2}| \leqslant \frac{\delta}{2}\frac{n^2 p}{2}} P_1(M) \cdot \mathrm{Prob}\big(\{G \in \mathscr{G}_{n,p} : e(G) = M\}\big)$$

and

$$P_2 = \sum_{|M - \frac{n^2 p}{2}| > \frac{\delta}{2}\frac{n^2 p}{2}} \mathrm{Prob}\big(\{G \in \mathscr{G}_{n,p} : e(G) = M\}\big).$$

Let $\mu > 0$ be given. We want to show that $P_1 < \frac{\mu}{2}$ and $P_2 < \frac{\mu}{2}$ for n large enough. The inequality $P_2 < \frac{\mu}{2}$ follows from the Chernoff bound (*cf.* Fact 5.5) on tails of the binomial distribution.

To bound P_1, we first consider the following chain of inequalities. Note that, because $Cn^{-1+1/l} \leqslant p(n) \leqslant cn^{-1+1/(l-1)}$, if M satisfies $|M - \frac{n^2 p}{2}| \leqslant \frac{\delta}{2}\frac{n^2 p}{2}$, then $\frac{1}{4}Cn^{1+1/l} \leqslant M \leqslant cn^{1+1/(l-1)}$. Hence, by (2.3), for M such that $|M - \frac{n^2 p}{2}| \leqslant \frac{\delta}{2}\frac{n^2 p}{2}$, we have

$$\begin{aligned}
P_1(M) &\leqslant \mathrm{Prob}\left(\left\{G \in \mathscr{G}_{n,p} : c(G) < \left(1-\frac{\delta}{2}\right)\frac{1}{l} \cdot M \mid e(G) = M\right\}\right) \\
&= \mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M) : c(G) < \left(1-\frac{\delta}{2}\right)\frac{1}{l} \cdot M\right\}\right) \\
&< \frac{\mu}{2}.
\end{aligned}$$

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Consequently

$$\begin{aligned} P_1 &= \sum_{|M-\frac{n^2p}{2}|\leqslant \frac{\delta}{2}\frac{n^2p}{2}} P_1(M)\cdot \mathrm{Prob}\big(\{G\in \mathscr{G}_{n,p} : e(G)=M\}\big) \\ &= \sum_{|M-\frac{n^2p}{2}|\leqslant \frac{\delta}{2}\frac{n^2p}{2}} P_1(M)\cdot \binom{\binom{n}{2}}{M} p^M (1-p)^{\binom{n}{2}-M} \\ &< \frac{\mu}{2}. \end{aligned}$$

□

Before giving the proof of Lemma 2.2 we need to introduce some tools. For real numbers $v>0$ and $\tau>0$, let $\mathscr{P}(\tau,v)$ be the following property of graphs.

We say that a graph G on n vertices and with M edges has property $\mathscr{P}(\tau,v)$ if and only if, for every subset of vertices A, $|A|\geqslant \tau n$, the number of edges $e(G[A])$ induced on A satisfies

$$(1-v)\binom{|A|}{2}\cdot\frac{M}{\binom{n}{2}} \leqslant e(G[A]) \leqslant (1+v)\binom{|A|}{2}\cdot\frac{M}{\binom{n}{2}}, \tag{2.4}$$

and for every pair of disjoint sets of vertices A and B, $|A|,|B|\geqslant \tau n$, the number of edges $e_G(A,B)$ between the sets A and B satisfies

$$(1-v)|A||B|\cdot\frac{M}{\binom{n}{2}} \leqslant e_G(A,B) \leqslant (1+v)|A||B|\cdot\frac{M}{\binom{n}{2}}. \tag{2.5}$$

Note that (2.4) implies (2.5) with v replaced by $v'=3v$. An elementary probabilistic argument, which we omit, yields the following fact.

Fact 2.3. *Let τ, v and a be positive real numbers. Then there exists a positive c such that, for all $M\geqslant cn$,*

$$\mathrm{Prob}\big(\{G\in\mathscr{G}(n,M) : G \text{ has property } \mathscr{P}(\tau,v)\}\big) = 1-O(n^{-a})$$

as $n\to\infty$.

We will be using the regularity lemma for sparse graphs in the proof of Lemma 2.2. Let $q,\tau, 0<q,\tau\leqslant 1$, and $\gamma>0$ be real numbers. A graph $H=(V,E)$ on $n=|V|$ vertices is (τ,q,γ)-*bounded* if, for any pair of disjoint sets $A,B\subseteq V$, $|A|\geqslant\tau n$, $|B|\geqslant\tau n$,

$$e_H(A,B)\leqslant \gamma q|A||B|.$$

For a (τ,q,γ)-bounded graph H, define the q-*density* $d_{H,q}(A,B)$ of a pair of disjoint nonempty sets $A,B\subseteq V$ by

$$d_{H,q}(A,B)=\frac{e_H(A,B)}{q|A||B|}.$$

Observe that for all pairs of disjoint nonempty sets $A,B\subseteq V$, $|A|,|B|\geqslant\tau n$, we have $d_{H,q}(A,B)\leqslant\gamma$.

Let $\epsilon>0$ be a real number. A pair of disjoint nonempty sets $A,B\subseteq V$ is called (ϵ,q)-*regular* if, for every $A'\subseteq A$, $B'\subseteq B$, $|A'|\geqslant\epsilon|A|$, $|B'|\geqslant\epsilon|B|$,

$$|d_{H,q}(A',B')-d_{H,q}(A,B)|<\epsilon.$$

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A partition $W_0, W_1, \ldots, W_t$ of the vertex set of the graph H is called an (ϵ, q)-*regular partition* if $|W_1| = \cdots = |W_t|$, $|W_0| < \epsilon n$, and all but at most $\epsilon\binom{t}{2}$ pairs W_i, W_j are (ϵ, q)-regular.

A regularity lemma for (τ, q, γ)-bounded graphs was independently considered by Kohayakawa and Rödl. For a survey see, for example, [8]. It will be sufficient for our purposes to formulate the regularity lemma just for the class of $(\tau, q, 1)$-bounded graphs.

Lemma 2.4. (Regularity lemma for sparse graphs). *For every $t_0 > 0$ and $\epsilon, 0 < \epsilon < 1$, there exist $T_0 = T_0(t_0, \epsilon) \geqslant t_0$, and $\tau = \tau(t_0, \epsilon, T_0)$, $0 < \tau \leqslant \frac{1}{T_0}$, and n_0 such that, for every $n \geqslant n_0$, the following holds. If*

(a) *H is a $(\tau, q, 1)$-bounded graph with any $0 < q \leqslant 1$, and*
(b) *$\mathcal{Q}$ is a partition of $V(H)$ into t_0 parts,*

then there is a refinement $\mathcal{Q}'$ of $\mathcal{Q}$ which is (ϵ, q)-regular and has $t = t_0 t_1, t \leqslant T_0$, partition classes. Moreover, if $T_0!$ divides the number of vertices of H, there exists an (ϵ, q)-regular refinement with the exceptional class empty.

A concept similar to that of $(h, m, K, \epsilon, q, \omega, D)$-configurations, which we are going to introduce below, was previously considered, for example, by Haxell, Łuczak and Kohayakawa [7], Kreuter [11], Kohayakawa, Łuczak and Rödl [10], and Łuczak [12]. We will follow a recent paper of Kohayakawa and Kreuter [9]. For our purposes it is enough to consider the density in property (c) below bounded from above by one. For a more general set-up see, *e.g.*, [9].

Let $0 < \epsilon, q, \omega \leqslant 1$ and $D \geqslant 1$ be real numbers and let m, K, and $h \geqslant 3$ be positive integers. Let $\mathcal{H}(h, m, K, \epsilon, q, \omega, D)$ denote the set of all h-partite graphs H with an h-partition $W_1, \ldots, W_h$ satisfying the following conditions (a)–(d).

(a) $|W_1| = \cdots = |W_h| = m$.
(b) $e(H) = K$ and all edges join vertices in consecutive classes W_i, *i.e.*, if e is an edge of H, then its endpoints belong to W_{i-1} and W_i, for some $1 \leqslant i \leqslant h$. (Here and below, the indices of the sets W_i will be taken modulo h.)
(c) The pairs (W_{i-1}, W_i), $1 \leqslant i \leqslant h$, are (ϵ, q)-regular and $\omega \leqslant d_{H,q}(W_{i-1}, W_i) \leqslant 1$, $1 \leqslant i \leqslant h$.

To state property (d) we need the following definition. We call a pair of disjoint sets of vertices U, U' in a graph $(h, \bar{d})$-*compatible* if $1 \leqslant |U| \leqslant |U'| \leqslant \bar{d}|U| \leqslant \bar{d}^{h-2}$. An $(h, \bar{d})$-compatible pair U, U' is D-*sparse* if $e(U, U') \leqslant D|U'|$. Now we are ready to state property (d).

(d) For all pairs (W_{i-1}, W_i), $2 \leqslant i < h$, every (h, qm)-compatible pair of sets $U \subseteq W_{i-1}$, $U' \subseteq W_i$ is D-sparse.

We call any graph satisfying (a)–(d) above an $(h, m, K, \epsilon, q, \omega, D)$-*configuration*. Note that properties (a)–(c) of an $(h, m, K, \epsilon, q, \omega, D)$-configuration imply that $\omega q m^2 h \leqslant K \leqslant q m^2 h$. Further, property (d) ensures that even the induced subgraphs between small sets U, U' (in the pairs (W_{i-1}, W_i), $2 \leqslant i < h$) for which regularity does not apply are not too dense.

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A cycle C_h on h vertices in an $(h, m, K, \epsilon, q, \omega, D)$-configuration is called a *principal* h-cycle if $|V(C_h) \cap W_i| = 1$ for all $i = 1, \ldots, h$.

We shall need the following lemma on $(h, m, K, \epsilon, q, \omega, D)$-configurations considered in a paper by Kohayakawa and Kreuter (*cf.* Lemma 17 in [9]). We would like to point out that Lemma 17 in [9] deals precisely with $(h, m, K, \epsilon, q, \omega, D)$-configurations without principal h-cycles, although as formulated in [9] it is a bit unclear which h-cycles are not allowed in the $(h, m, K, \epsilon, q, \omega, D)$-configurations counted.

Lemma 2.5. *Let an integer $h \geqslant 3$ and constants α and ω, $0 < \alpha, \omega \leqslant 1$, and $D \geqslant 1$ be given. Then there exist $B_0 = B_0(h, \alpha, \omega, D)$, $m_0 = m_0(h, \alpha, \omega, D)$ and $\epsilon_0 = \epsilon_0(h, \alpha, \omega, D) > 0$ such that, for every $m \geqslant m_0$, $B \geqslant B_0$, and $K \geqslant h\omega B m^{1+\frac{1}{h-1}}$ the number of $(h, m, K, \epsilon_0, Bm^{-1+\frac{1}{h-1}}, \omega, D)$-configurations which do not contain a principal h-cycle is at most*

$$\alpha^K \cdot \binom{(h+2)m^2}{K}.$$

Let us point out the following observation concerning Lemma 2.5. Let $0 < \alpha \leqslant 1$ and let ϵ_0 be given by Lemma 2.5. Since, for all $0 < \epsilon \leqslant \epsilon_0$, (ϵ, q)-regularity implies (ϵ_0, q)-regularity, the conclusion of Lemma 2.5 is valid for $(h, m, K, \epsilon, Bm^{-1+\frac{1}{h-1}}, \omega, D)$-configurations whenever $0 < \epsilon \leqslant \epsilon_0$.

Let H be a graph satisfying (a)–(d) above (*i.e.*, H is an $(h, m, K, \epsilon, q, \omega, D)$-configuration) and

(e) H does not contain a principal h-cycle.

We will call such a graph H a *rare* $(h, m, K, \epsilon, q, \omega, D)$-*configuration.*

We say that a graph G has *property* $\mathscr{R}(h, \bar{d}, D)$ if every $(h, \bar{d})$-compatible pair $U, U' \subseteq V(G)$ is D-sparse.

Claim 2.6. *Let an integer $l \geqslant 2$ and real numbers $\bar{B}, c_1, c_2 > 0$ be given. Let n be a positive integer. For M such that $c_1 n^{1+1/l} \leqslant M \leqslant c_2 n^{1+1/l}$,*

$$\mathrm{Prob}\big(\{G \in \mathscr{G}(n, M) : G \text{ has property } \mathscr{R}(l+1, \bar{B}n^{1/l}, 6l)\}\big) = 1 - o(n^{-1}) \tag{2.6}$$

as $n \to \infty$.

Proof. Set $\bar{d} = \bar{B}n^{1/l}$, $D = 6l$, and $\bar{p} = M/\binom{n}{2}$. We start the proof by showing that

$$\mathrm{Prob}\big(\{G \in \mathscr{G}_{n,\bar{p}} : G \text{ has property } \mathscr{R}(l+1, \bar{d}, D)\}\big) \geqslant 1 - n^{-3}$$

for n sufficiently large.

First assume $l \geqslant 3$. For a fixed $(l+1, \bar{d})$-compatible pair U, U',

$$\mathrm{Prob}\big(\{G \in \mathscr{G}_{n,\bar{p}} : e(U, U') > D|U'|\}\big) \leqslant \binom{|U||U'|}{D|U'|+1} \bar{p}^{D|U'|+1}.$$

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(Note that for a binomially distributed random variable $X \sim \mathrm{Bi}(m,p)$ we have $\mathrm{Prob}(X \geqslant k) \leqslant \binom{m}{k}p^k$.) Since $1 \leqslant |U| \leqslant |U'| \leqslant \bar{d}|U| \leqslant \bar{d}^{l-1}$, we get

$$\begin{aligned}
&\mathrm{Prob}\big(\{G \in \mathscr{G}_{n,\bar{p}} : G \text{ does not have } \mathscr{R}(l+1,\bar{d},D)\}\big)\\
&\quad\leqslant \sum_{\substack{U,U':\ \text{the pair } U,U' \text{ is}\\ (l+1,\bar{d})\text{-compatible}}} \mathrm{Prob}\big(\{G \in \mathscr{G}_{n,\bar{p}} : e(U,U') > D|U'|\}\big)\\
&\quad\leqslant \sum_{|U|=1}^{\lfloor \bar{d}^{l-2}\rfloor} \sum_{|U'|=|U|}^{\lfloor \bar{d}|U|\rfloor} \binom{n}{|U|}\binom{n}{|U'|}\binom{|U||U'|}{D|U'|+1}\bar{p}^{D|U'|+1}\\
&\quad\leqslant \sum_{|U|=1}^{\lfloor \bar{d}^{l-2}\rfloor} \sum_{|U'|=|U|}^{\lfloor \bar{d}|U|\rfloor} \binom{n}{|U'|}^2\binom{|U||U'|}{D|U'|+1}\bar{p}^{D|U'|+1}\\
&\quad\leqslant \sum_{|U|=1}^{\lfloor \bar{d}^{l-2}\rfloor} \sum_{|U'|=|U|}^{\lfloor \bar{d}|U|\rfloor} \frac{1}{|U'|^{2|U'|}}\cdot\frac{e|U||U'|}{D|U'|+1}\cdot\bar{p}\cdot\left(e^2n^2\cdot\left(\frac{e|U||U'|}{D|U'|+1}\right)^D\bar{p}^D\right)^{|U'|}\\
&\quad\leqslant \sum_{|U|=1}^{\lfloor \bar{d}^{l-2}\rfloor} \sum_{|U'|=|U|}^{\lfloor \bar{d}|U|\rfloor} \frac{|U|}{|U|^{2|U|}}\cdot\frac{e}{D}\cdot\bar{p}\left(\frac{e^{D+2}}{D^D}\cdot n^2(|U|\bar{p})^D\right)^{|U'|}. \qquad (2.7)
\end{aligned}$$

The expression in (2.7) can be further bounded from above using $\bar{p} \leqslant c_2 n^{1+1/l}/\binom{n}{2} \leqslant 3c_2 n^{-1+1/l}$, $D = 6l$, and $|U| \leqslant \bar{d}^{l-2}$ where $\bar{d} = \bar{B}n^{1/l}$. Thus we have

$$\begin{aligned}
&\mathrm{Prob}\big(\{G \in \mathscr{G}_{n,\bar{p}} : G \text{ does not have } \mathscr{R}(l+1,\bar{d},D)\}\big)\\
&\quad\leqslant \sum_{|U|=1}^{\lfloor \bar{d}^{l-2}\rfloor} \sum_{|U'|=|U|}^{\lfloor \bar{d}|U|\rfloor} \frac{|U|}{|U|^{2|U|}}\cdot\frac{e}{D}\cdot\bar{p}\left(\frac{e^{D+2}}{D^D}\cdot(3c_2\bar{B}^{l-2})^D\cdot n^{-4}\right)^{|U'|}\\
&\quad\leqslant \sum_{|U|=1}^{\lfloor \bar{d}^{l-2}\rfloor} \bar{d}|U|\frac{|U|}{|U|^{2|U|}}\cdot\frac{e}{D}\cdot\bar{p}\left(\frac{e^{D+2}}{D^D}\cdot(3c_2\bar{B}^{l-2})^D\cdot n^{-4}\right)\\
&\quad\leqslant \sum_{|U|=1}^{\lfloor \bar{d}^{l-2}\rfloor} \bar{d}\bar{p}\cdot\frac{e^{D+3}}{D^{D+1}}\cdot(3c_2\bar{B}^{l-2})^D\cdot n^{-4}\\
&\quad\leqslant \bar{d}^{l-1}\bar{p}\cdot\frac{e^{D+3}}{D^{D+1}}\cdot(3c_2\bar{B}^{l-2})^D\cdot n^{-4}\\
&\quad\leqslant \bar{B}^{l-1}n^{\frac{l-1}{l}}\cdot 3c_2 n^{-1+1/l}\cdot\frac{e^{D+3}}{D^{D+1}}\cdot(3c_2\bar{B}^{l-2})^D\cdot n^{-4}\\
&\quad\leqslant n^{-3}
\end{aligned}$$

for n sufficiently large.

Now let $l = 2$ and let U, U' be two subsets of vertices. We observe that the inequalities $1 \leqslant |U| \leqslant |U'| \leqslant \bar{d}|U| \leqslant \bar{d}^{l-1}$ imply $|U| = 1$. In this case the trivial inequality $e(U,U') \leqslant |U'|$ implies $\mathrm{Prob}(\{G \in \mathscr{G}_{n,\bar{p}} : e(U,U') > D|U'|\}) = 0$. Thus $\mathrm{Prob}(\{G \in \mathscr{G}_{n,\bar{p}} : G \text{ does not have } \mathscr{R}(l+1,\bar{d},D)\}) = 0$.

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We use the standard inequality for changing from the binomial to the uniform model (*cf.* [3] p. 35) to conclude that

$$\begin{aligned}
&\mathrm{Prob}\big(\{G \in \mathscr{G}(n,M) : G \text{ does not have } \mathscr{R}(l+1,\bar{d},D)\}\big)\\
&\quad\leqslant 3\sqrt{M}\,\mathrm{Prob}\left(\left\{\mathscr{G}_{n,\frac{M}{\binom{n}{2}}} \text{ does not have } \mathscr{R}(l+1,\bar{d},D)\right\}\right)\\
&\quad\leqslant 3n\cdot n^{-3}\\
&\quad= o(n^{-1}).
\end{aligned}$$

□

Proof of Lemma 2.2. Let $l \geqslant 2$ and $\delta > 0$ be given. Before giving the proof we need to choose some constants. Set

$$\omega = \frac{\delta}{16l} \quad \text{and} \quad \alpha = \frac{\delta}{128el}. \tag{2.8}$$

Let $\epsilon_0 = \epsilon_0(l+1,\alpha,\omega,6l)$ be given by Lemma 2.5. We define

$$\epsilon = \min\left\{\frac{\delta}{8l}, \epsilon_0\right\} \tag{2.9}$$

and

$$t_0 = \lceil 1/\epsilon \rceil. \tag{2.10}$$

Further, we set

$$T_0 = T_0(\epsilon, t_0), \quad T_0 \geqslant t_0, \tag{2.11}$$

to be the constant which exists by Lemma 2.4 and

$$B_0 = B_0(l+1,\alpha,\omega,6l), \tag{2.12}$$

where $B_0(l+1,\alpha,\omega,6l)$ is given by Lemma 2.5. Set

$$c_1 = \frac{1}{8}\cdot T_0 B_0 \quad \text{and} \quad c_2 = 2c_1. \tag{2.13}$$

Further, let

$$\bar{B} = \frac{4c_2}{t_0}. \tag{2.14}$$

Note that $\bar{B} \geqslant B_0$. Set

$$q = 4c_2 n^{-1+1/l}. \tag{2.15}$$

Finally, let M satisfy $c_1 n^{1+1/l} \leqslant M \leqslant c_2 n^{1+1/l}$.

We shall need the following claim in the proof.

Claim 2.7. *Let $l \geqslant 2$ be an integer and δ, $0 < \delta < 1$, be a real number. Let ω, ϵ, t_0, T_0, B_0, $\bar{B}$, c_1, c_2, and q be given by* (2.8)–(2.15). *Then there exist positive numbers $v = v(l,\delta)$, $\tau = \tau(t_0,\epsilon,T_0)$ and n_0 such that, for $n \geqslant n_0$, every graph G satisfying*

(i) *G is a graph on n vertices having M edges, $c_1 n^{1+1/l} \leqslant M \leqslant c_2 n^{1+1/l}$,*

(ii) *G has property $\mathscr{P}(\tau, v)$,*

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(iii) *G has property* $\mathscr{R}(l+1, \bar{B}n^{1/l}, 6l)$, *and*
(iv) $c(G) < (1-\delta)\frac{1}{t} \cdot M$

contains a rare $(l+1, m, K, \epsilon, q, \omega, 6l)$*-configuration with* $m = \frac{n}{t}$, *and* $K \geqslant (l+1)\frac{\delta}{4l} \cdot \frac{1}{t^{1-1/l}} \cdot c_2 m^{1+1/l}$ *for some* t *satisfying* $t_0 \leqslant t \leqslant T_0$.

We defer the proof of Claim 2.7 to the end of this section.

Let $v = v(l, \delta)$ and $\tau = \tau(t_0, \epsilon, T_0)$ be given by Claim 2.7. Note that due to assumptions of Lemma 2.2 every graph $G \in \mathscr{G}(n, M)$ satisfies (i) of Claim 2.7. By Fact 2.3 and Claim 2.6

$$\mathrm{Prob}\big(\{G \in \mathscr{G}(n, M) : G \text{ has properties } \mathscr{P}(\tau, v) \text{ and } \mathscr{R}(l+1, \bar{B}n^{1/l}, 6l)\}\big) = 1 - o(n^{-1}).$$

Let $G \in \mathscr{G}(n, M)$ be a graph satisfying (i), (ii), (iii), and (iv) of Claim 2.7. By Claim 2.7 graph G contains a rare $(l+1, m, K, \epsilon, q, \omega, 6l)$-configuration with $m = \frac{n}{t}$. Here t satisfies $t_0 \leqslant t \leqslant T_0$. Our aim is to show that such a configuration in $G \in \mathscr{G}(n, M)$ is very unlikely, namely that it appears with probability $o(n^{-1})$.

Let λ be the number of rare $(l+1, m, K, \epsilon, q, \omega, 6l)$-configurations in the complete graph on n vertices. First we give an upper bound on λ. Let $W_1, \ldots, W_{l+1}$ be fixed disjoint sets of vertices such that $|W_i| = m$, $1 \leqslant i \leqslant l+1$. There are at most $\binom{(l+1)m^2}{K}$ $(l+1, m, K, \epsilon, q, \omega, 6l)$-configurations on these vertex sets. Since $q = Bm^{-1+1/l}$, where $B = \frac{4c_2}{t^{1-1/l}} \geqslant \frac{4c_2}{T_0} = B_0$ and $K \geqslant (l+1) \cdot \frac{\delta}{4l} \cdot \frac{1}{t^{1-1/l}} \cdot c_2 m^{1+1/l} = (l+1)\omega q m^2$, we can apply Lemma 2.5 and conclude that at most $\alpha^K \binom{(l+3)m^2}{K}$ of these $(l+1, m, K, \epsilon, q, \omega, 6l)$-configurations are rare. Hence

$$\lambda \leqslant \binom{n}{m}^{l+1} \cdot \alpha^K \cdot \binom{(l+3)m^2}{K} \leqslant \left(\frac{en}{m}\right)^{m(l+1)} \cdot \left(\frac{e(l+3)m^2}{K} \cdot \alpha\right)^K.$$

Let $\Lambda_1, \ldots, \Lambda_\lambda$ be an enumeration of rare $(l+1, m, K, \epsilon, q, \omega, 6l)$-configurations in the complete graph on n vertices. Let $X_v = X_{\Lambda_v}(G)$, $1 \leqslant v \leqslant \lambda$, and $G \in \mathscr{G}(n, M)$ be random indicator variables defined by $X_v = 1$ if Λ_v is a subgraph of G and $X_v = 0$ otherwise. For all $1 \leqslant v \leqslant \lambda$,

$$\mathrm{Prob}\big(X_v = 1\big) = \frac{\binom{\binom{n}{2}-K}{M-K}}{\binom{\binom{n}{2}}{M}} \leqslant \left(\frac{M}{\binom{n}{2} - K}\right)^K.$$

Finally, set $X = \sum_{v=1}^{\lambda} X_v$. Recall that $n = mt$, $K \geqslant (l+1) \cdot \frac{\delta}{4l} \cdot \frac{1}{t^{1-1/l}} \cdot c_2 m^{1+1/l}$, and $T_0 \geqslant t \geqslant t_0 \geqslant 8l$. Hence

$$\begin{aligned}
&\mathrm{Prob}\big(\{G \in \mathscr{G}(n, M) : G \text{ contains a rare } (l+1, m, K, \epsilon, q, \omega, 6l)\text{-configuration}\} \\
&\qquad\qquad \mid \{G \in \mathscr{G}(n, M) : G \text{ has } \mathscr{P}(\tau, v) \text{ and } \mathscr{R}(l+1, \bar{B}n^{1/l}, 6l)\}\big) \\
&= \mathrm{Prob}\big(X > 0 \mid \{G \in \mathscr{G}(n, M) : G \text{ has } \mathscr{P}(\tau, v) \text{ and } \mathscr{R}(l+1, \bar{B}n^{1/l}, 6l)\}\big) \\
&\leqslant 2\,\mathrm{Prob}\big(X > 0\big).
\end{aligned}$$

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The last inequality follows from the fact that $\mathrm{Prob}(\{G \in \mathscr{G}(n,M) : G \text{ has } \mathscr{P}(\tau,v) \text{ and } \mathscr{R}(l+1,\bar{B}n^{1/l},6l)\}) = 1 - o(n^{-1})$. Hence, we need to bound $\mathrm{Prob}(X > 0)$:

$$\begin{aligned}
\mathrm{Prob}\big(X>0\big) &\leqslant \mathbf{E}\big[X\big] \\
&\leqslant \left(\frac{en}{m}\right)^{m(l+1)} \cdot \left(\frac{e(l+3)m^2}{K}\cdot\alpha\right)^K \cdot \left(\frac{M}{\binom{n}{2}-K}\right)^K \\
&\leqslant \left(\frac{en}{m}\right)^{m(l+1)} \cdot \left(e(l+3)\alpha\cdot\frac{M}{K}\cdot\frac{m^2}{\binom{n}{2}-K}\right)^K \\
&\leqslant \left(\frac{en}{m}\right)^{m(l+1)} \cdot \left(e(l+3)\alpha\cdot\frac{t^{1+1/l}}{(l+1)\frac{\delta}{4l}t^{-1+1/l}}\cdot\frac{1}{\frac{t^2}{4}-(l+1)}\right)^K \\
&\leqslant \left(\frac{en}{m}\right)^{m(l+1)} \cdot \left(\frac{4el}{\delta}\cdot 2\cdot\alpha\cdot\frac{t^2}{\frac{t^2}{4}-(l+1)}\right)^K \\
&\leqslant \left(\frac{en}{m}\right)^{m(l+1)} \cdot \left(\frac{64el}{\delta}\cdot\alpha\right)^K \\
&\leqslant \left(\frac{en}{m}\right)^{m(l+1)} \cdot 2^{-K} \\
&= o(n^{-1}).
\end{aligned}$$

Our bounds above imply

$$\begin{aligned}
\mathrm{Prob}\big(\{G \in \mathscr{G}(n,M) : G \text{ contains a rare } (l+1,m,K,\epsilon,q,\omega,6l)\text{-configuration}\} \\
\mid \{G \in \mathscr{G}(n,M) : G \text{ has } \mathscr{P}(\tau,v) \text{ and } \mathscr{R}(l+1,\bar{B}n^{1/l},6l)\}\big) = o(n^{-1}).
\end{aligned}$$

Summarizing, we have

$$\begin{aligned}
&\mathrm{Prob}\left(\left\{G \in \mathscr{G}(n,M) : c(G) < (1-\delta)\cdot\frac{1}{l}\cdot M\right\}\right) \\
&\quad< \mathrm{Prob}\big(\{G \in \mathscr{G}(n,M) : G \text{ does not have both } \mathscr{P}(\tau,v) \text{ and } \mathscr{R}(l+1,\bar{B}n^{1/l},6l)\}\big) \\
&\qquad+ \mathrm{Prob}\big(\{G \in \mathscr{G}(n,M) : G \text{ contains a rare } (l+1,m,K,\epsilon,q,\omega,6l)\text{-configuration}\} \\
&\qquad\qquad \mid \{G \in \mathscr{G}(n,M) : G \text{ has } \mathscr{P}(\tau,v) \text{ and } \mathscr{R}(l+1,\bar{B}n^{1/l},6l)\}\big) \\
&\quad= o(n^{-1}).
\end{aligned}$$

□

Proof of Claim 2.7. Let $l \geqslant 2$, and $\delta > 0$ be given. Recall that we set ω, ϵ, t_0, T_0, B_0, $\bar{B}$, c_1, c_2, and q in (2.8)–(2.15). Further, let $v = \frac{\delta}{8(l-1)}$ and let $\tau = \tau(t_0,\epsilon,T_0)$ be given by Lemma 2.4. We assume that n is large enough.

Consider a graph G satisfying (i)–(iv). Owing to our assumption $c(G) < (1-\delta)\frac{1}{l}\cdot M$ there is a subset of edges $F \subseteq E(G)$ such that $G \setminus F$ is a cover graph and $|F| < (1-\delta)\frac{1}{l}\cdot M$. Set $\bar{G} = G \setminus F$. Thus,

$$e(\bar{G}) = e(G) - |F| > M - (1-\delta)\frac{1}{l}\cdot M = M\cdot\left(1 - \frac{1}{l} + \frac{\delta}{l}\right).$$

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Since $\bar{G}$ is a cover graph, there exists an acyclic orientation $\vec{G}$ of $\bar{G}$ which does not contain any dependent arcs. Let $\prec$ be a linear order on $V(\bar{G})$ containing all arcs of $\vec{G}$. For two sets $A, B \subseteq V(\bar{G})$, we write $A \prec B$ if and only if $\max A \prec \min B$.

Since we are interested in an asymptotic result, we may assume that ϵn and $t_0 = 1/\epsilon$ are integers and hence we will drop the integer part signs in our computations. Similarly, we will assume that t_0 divides n.

Consider a partition $\cup_{i=1}^{t_0} A_i$ of $V(\bar{G})$ into t_0 pairwise disjoint intervals each of cardinality ϵn, such that $A_1 \prec A_2 \prec \cdots \prec A_{t_0}$. We delete all the edges of $\bar{G}$ induced on sets $A_1, \ldots, A_{t_0}$ and call $\tilde{G}$ the resulting graph. By (ii), $e_{\bar{G}}(\bar{G}[A_i]) \leqslant 2\epsilon^2 M$ for all $i = 1, 2, \ldots, t_0$, and hence

$$\begin{aligned} e(\tilde{G}) &= e(\bar{G}) - \sum_{i=1}^{t_0} e_{\bar{G}}(\bar{G}[A_i]) \\ &\geqslant e(\bar{G}) - t_0 \cdot 2\epsilon^2 M \\ &\geqslant M \cdot \left(1 - \frac{1}{l} + \frac{3}{4} \cdot \frac{\delta}{l}\right). \end{aligned}$$

Owing to property $\mathscr{P}(\tau, \nu)$ the graph $\tilde{G}$ is $(\tau, q, 1)$-bounded. We apply the regularity lemma for sparse graphs (Lemma 2.4) to the graph $\tilde{G}$ with the initial partition $\mathscr{Q} = \{A_1, \ldots, A_{t_0}\}$. Let $\mathscr{Q}' = \{B_{i,j}\}_{(i,j) \in [t_0] \times [t_1]}$ be the (ϵ, q)-regular refinement of the partition $\mathscr{Q}$ given by Lemma 2.4. We may assume that $A_i = \cup_{j \in [t_1]} B_{i,j}$, $i = 1, \ldots, t_0$. Thus $|B_{i,j}| = \frac{n}{t}$, $i = 1, \ldots, t_0$, $j = 1, \ldots, t_1$, where $t = t_0 t_1$.

A pair of partition classes $S_1, S_2 \in \mathscr{Q}'$ is called *sparse* if its q-density is less than ω. In particular, this means that the number of edges between S_1 and S_2 is less than $\omega q \left(\frac{n}{t}\right)^2 = \frac{\delta}{2l} \cdot \frac{c_1}{t^2} \cdot n^{1+1/l} \leqslant \frac{\delta}{2l} \cdot \frac{M}{t^2}$ edges. Otherwise, we call the pair *dense*.

Let H be the graph obtained from $\tilde{G}$ by deleting all edges of irregular and sparse pairs $(B_{i,j}, B_{i',j'})$. Using property $\mathscr{P}(\tau, \nu)$ and the fact that there are at most $\epsilon \binom{t}{2}$ irregular pairs, we infer that the number of edges in irregular pairs is at most

$$\epsilon \binom{t}{2} (1 + \nu) \left(\frac{n}{t}\right)^2 \frac{M}{\binom{n}{2}} \leqslant 2\epsilon M \leqslant \frac{\delta}{4l} \cdot M.$$

Further, the number of edges in sparse pairs is at most

$$\binom{t}{2} \frac{\delta}{2l} \cdot \frac{M}{t^2} < \frac{\delta}{4l} M.$$

Hence, the number of edges in graph H is

$$\begin{aligned} e(H) &> M \cdot \left(1 - \frac{1}{l} + \frac{3}{4} \cdot \frac{\delta}{l}\right) - \frac{\delta}{4l} M - \frac{\delta}{4l} M \\ &= M \cdot \left(1 - \frac{1}{l} + \frac{1}{4} \cdot \frac{\delta}{l}\right). \end{aligned}$$

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Owing to property $\mathscr{P}(\tau, \nu)$ there are at most $(1+\nu)\left(\frac{n}{t}\right)^2 \frac{M}{\binom{n}{2}}$ edges between each pair of sets $S_1, S_2 \in \mathscr{Q}'$ and thus we infer that there are at least

$$\frac{e(H)}{(1+\nu)\left(\frac{n}{t}\right)^2 \frac{M}{\binom{n}{2}}} > \frac{1-\frac{1}{l}+\frac{\delta}{4l}}{1+\nu} \cdot \frac{n-1}{n} \cdot \frac{t^2}{2} \geqslant \left(1-\frac{1}{l}\right)\frac{t^2}{2}$$

pairs of partition sets in H which are both dense and (ϵ, q)-regular. Turán's theorem implies that there are partition sets $C_k = B_{i_k, j_k}$, $1 \leqslant k \leqslant l+1$, such that all $\binom{l+1}{2}$ pairs $C_k, C_{k'}$ are dense and (ϵ, q)-regular. We may assume that $i_1 < i_2 < \cdots < i_{l+1}$ and therefore also $C_1 \prec C_2 \prec \cdots \prec C_{l+1}$, where $\prec$ is the linear order defined above. Let $F_k, k = 1, \ldots, l$, be the bipartite subgraphs of H induced on bipartition (C_k, C_{k+1}). Similarly, let F_{l+1} be the bipartite subgraph of H induced on bipartition (C_1, C_{l+1}). Set $F = \cup_{k=1}^{l+1} F_k$ and $K = e(F)$.

Note that F contains no principal $(l+1)$-cycle. For a contradiction, assume that F contains a principal $(l+1)$-cycle with vertex set $\{u_1, \ldots, u_{l+1}\}$ where $u_k \in C_k$. Since $u_k \in C_k \subseteq A_{i_k}$, this implies that there is a directed cycle in $\vec{G}$ with the vertex set $\{u_1, \ldots, u_{l+1}\}$, *i.e.*, the arcs $(u_1, u_2), \ldots, (u_l, u_{l+1}), (u_1, u_{l+1})$ belong to $\vec{G}$. This, however, is a contradiction to the choice of $\vec{G}$.

Since all pairs (C_1, C_{l+1}) and (C_i, C_{i+1}), $1 \leqslant i \leqslant l$, in F have density at least ω, each of them contains at least $\omega q\left(\frac{n}{t}\right)^2 = \frac{\delta}{4l} \cdot \frac{c_2}{t^2} \cdot n^{1+1/l}$ edges. Thus $K \geqslant (l+1)\frac{\delta}{4l} \cdot \frac{c_2}{t^2} \cdot n^{1+1/l}$.

In order to show F is a rare $(l+1, \frac{n}{t}, K, \epsilon, q, \omega, 6l)$-configuration, we still need to show that F has property $\mathscr{R}(l+1, \frac{4c_2}{t} n^{1/l}, 6l)$. Since F is a subgraph of G, F has property $\mathscr{R}(l+1, \bar{B} n^{1/l}, 6l)$. Further, since $\bar{B} = \frac{4c_2}{t_0} \geqslant \frac{4c_2}{t}$, every $(l+1, \bar{B} n^{1/l})$-compatible pair of sets of vertices in F is also $(l+1, \frac{4c_2}{t} n^{1/l})$-compatible. Thus, F has property $\mathscr{R}(l+1, \frac{4c_2}{t} n^{1/l}, 6l)$. This finishes the proof of Claim 2.7. □

3. Upper bounds on $r_{\chi,g}$

In this section we include a proof of Theorem 1.4 in the case that $\chi < g$. Further, we show an upper bound on $r_{\chi,g}$ in the case that $\chi \geqslant g$.

For the sake of completeness we include the well-known proof of Theorem 1.4 for the case $\chi < g$.

Proof of Theorem 1.4. The case $\chi < g$: Let G be a graph with $\chi(G) = \chi$, $\mathrm{girth}(G) = g$, and $\chi < g$. Let $V_1, \ldots, V_\chi$ be the colour classes of a χ-colouring of G. We define the following orientation $\vec{G}$ of G. Let $\{u, v\}$ be an edge of G such that $u \in V_i$ and $v \in V_j$ with $i < j$. We define $(u, v) \in \vec{G}$, *i.e.*, we orient the edge $\{u, v\}$ from u to v. Clearly, $\vec{G}$ is acyclic and, as $\mathrm{girth}(G) = g > \chi$, $\vec{G}$ does not contain dependent arcs. Thus, $d_{\min}(G) = 0$. □

The proof of the upper bound for (1.7) in the case that $\chi \geqslant g$ immediately follows from the following proposition.

Proposition 3.1. *Let χ and g be integers satisfying $\chi \geqslant g \geqslant 3$. Then, for every graph G with $\chi(G) = \chi$ and $\mathrm{girth}(G) = g$,*

$$d_{\min}(G) \leqslant \frac{\binom{\chi-g+2}{2}}{\binom{\chi}{2}} \cdot e(G).$$

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Proof. Let G be a graph with chromatic number χ and girth g. Consider a χ-colouring $V_1 \cup \cdots \cup V_\chi$ of G.

Let e_{jk} denote the number of edges between V_j and V_k, $1 \leqslant j, k \leqslant \chi$. Clearly, $e_{jk} = e_{kj}$ and $e_{jj} = 0$.

For a permutation π of $[\chi]$, we define

$$\tilde{e}(\pi) = \sum_{j=1}^{g-2} \sum_{i=1}^{\chi-j} e_{\pi(i)\pi(i+j)}.$$

When summing up $\tilde{e}(\pi)$ over all permutations, we count each edge $2(\chi-j)(\chi-2)!$ times in the $\sum_{i=1}^{\chi-j} e_{\pi(i)\pi(i+j)}$ term, for every $j \in [g-2]$. Thus, we get

$$\begin{aligned}\sum_{\pi \text{ a permutation of } [\chi]} \tilde{e}(\pi) &= \big[2(\chi-1)(\chi-2)! + 2(\chi-2)(\chi-2)! + \cdots + \\ &\quad + 2(\chi-(g-2))(\chi-2)!\big] e(G) \\ &= 2(\chi-2)!\big[(\chi-1) + \cdots + (\chi-g+2)\big] e(G) \\ &= (g-2)(2\chi-g+1)(\chi-2)! \cdot e(G).\end{aligned}$$

By averaging, there is a permutation π of $[\chi]$ such that

$$\tilde{e}(\pi) \geqslant \frac{(g-2)(2\chi-g+1)}{\chi(\chi-1)} e(G).$$

We fix this permutation π. Using π, we define the following orientation $\vec{G}$ of G. Let $\{u,v\}$ be an edge of G such that $u \in V_{\pi(i)}$ and $v \in V_{\pi(j)}$ with $i < j$. We define $(u,v) \in \vec{G}$, *i.e.*, we orient the edge $\{u,v\}$ from u to v. Clearly, $\vec{G}$ is acyclic. Moreover, reversing the orientation of an arc $(u,v) \in \vec{G}$, with $u \in V_{\pi(i)}$, $v \in V_{\pi(j)}$, and $1 \leqslant j-i \leqslant g-2$ does not create an oriented cycle since $\mathrm{girth}(G) = g$. Thus, $\vec{G}$ contains at least $\tilde{e}(\pi)$ arcs which are not dependent. Hence,

$$\begin{aligned}d_{\min}(G) &\leqslant e(G) - \tilde{e}(\pi) \\ &\leqslant e(G) - \frac{(g-2)(2\chi-g+1)}{\chi(\chi-1)} e(G) \\ &= \frac{\binom{\chi-g+2}{2}}{\binom{\chi}{2}} e(G).\end{aligned}$$

□

4. Lower bounds on $r_{\chi,g}$

In this section we show how the lower bound in Theorem 1.4 in the case that $\chi \geqslant g$ follows from Proposition 4.1, which we state in this section. Proposition 4.1 contains the main part of the proof of the lower bound for $r_{\chi,g}$ in the case that $\chi \geqslant g$.

First, let us consider a parameter similar to $r_{\chi,g}$. Namely, set

$$r_{\chi,\geqslant g} = \sup_G \left\{ \frac{d_{\min}(G)}{e(G)} : \chi(G) = \chi,\ \mathrm{girth}(G) \geqslant g \right\}. \tag{4.1}$$

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Proposition 4.1 deals with a lower bound on $r_{\chi,\geqslant g}$. Its proof is deferred to Section 8.

Proposition 4.1. *Let χ and g be integers satisfying $\chi \geqslant g \geqslant 3$. For every sufficiently small $\epsilon > 0$ and for every sufficiently large m with $m \equiv 0 \pmod{\chi}$, there exists a graph G on m vertices satisfying*

(a) $\mathrm{girth}(G) \geqslant g, \chi(G) = \chi$, *and*
(b) *for every acyclic orientation $\vec{G}$ of G, the number $d(\vec{G})$ of dependent arcs in $\vec{G}$ satisfies*

$$d(\vec{G}) \geqslant \frac{\binom{\chi-g+2}{2}}{\binom{\chi}{2}} \cdot e(G)(1-\epsilon).$$

The following lemma shows that, in fact, $r_{\chi,g}$ and $r_{\chi,\geqslant g}$ are equal to each other.

Lemma 4.2. *For every two integers χ and g, such that $\chi \geqslant g \geqslant 3$, we have*

$$r_{\chi,g} = r_{\chi,\geqslant g}.$$

Proof. The equivalent definition of $r_{\chi,\geqslant g}$ is $r_{\chi,\geqslant g} = \sup_{h \geqslant g} r_{\chi,h}$ (*cf.* (4.1)). Clearly, $r_{\chi,\geqslant g} = \max\{r_{\chi,g}, \sup_{h\geqslant g+1} r_{\chi,h}\}$. We proved in Section 3 that $r_{\chi,h} \leqslant \binom{\chi-h+2}{2}/\binom{\chi}{2}$ for every h. Hence, $\sup_{h\geqslant g+1} r_{\chi,h} \leqslant \binom{\chi-g+1}{2}/\binom{\chi}{2}$.

On the other hand, Proposition 4.1 implies $r_{\chi,\geqslant g} \geqslant \binom{\chi-g+2}{2}/\binom{\chi}{2}$. Since $\binom{\chi-g+1}{2}/\binom{\chi}{2} < \binom{\chi-g+2}{2}/\binom{\chi}{2}$, we must have $r_{\chi,\geqslant g} = \max\{r_{\chi,g}, \sup_{h\geqslant g+1} r_{\chi,h}\} = r_{\chi,g}$. □

Proof of Theorem 1.4. The case $\chi \geqslant g$: Combine Proposition 3.1, Proposition 4.1, and Lemma 4.2. □

5. A probabilistic construction

In this section we show the existence of graphs which are used in the proof of the lower bound on $r_{\chi,\geqslant g}$ in Proposition 4.1. The probabilistic construction is contained in Proposition 5.1. We start with some necessary definitions.

The key property used in the proof of Proposition 4.1 is described in the following definition.

Definition 3. Let χ and g, $\chi \geqslant g \geqslant 3$, be positive integers. Let G be a χ-partite graph on m vertices with $V(G) = V_1 \cup \cdots \cup V_\chi$.

(i) Let an integer $r, g \leqslant r \leqslant \chi$, and the sets $W_1, \ldots, W_r, W_i \subseteq V_{i_j}$ for some $i_j \in \{1, \ldots, \chi\}$ (each W_i belongs to a different set V_{i_j}), $i = 1, \ldots, r$, be given. We say that the edge $\{u, v\} \in G[W_1 \cup W_r]$, $u \in W_1$, $v \in W_r$, is $(W_j)_{j=2}^{r-1}$-*connected* if there exist a $k \geqslant g$, indices $h_1, \ldots, h_k \in \{1, \ldots, r\}$, $h_1 = 1 < h_2 < \cdots < h_k = r$, and a path $w_{h_1}, w_{h_2}, \ldots, w_{h_k}$ in $G[W_{h_1} \cup W_{h_2} \cup \cdots \cup W_{h_k}]$ such that $w_{h_1} = u$, $w_{h_k} = v$, and $w_{h_j} \in W_{h_j}$ for all $j = 1, \ldots, k$.

If the edge $\{u, v\} \in G[W_1 \cup W_r]$ is not $(W_j)_{j=2}^{r-1}$-connected we call it $(W_j)_{j=2}^{r-1}$-*disconnected*.

(ii) Let $\beta, 0 < \beta < 1$, be a real number. For $\psi > 0$, possibly a function of m, we say that the graph G has the (β, ψ)-*property* if, for every choice of sets $W_{i_1}, \ldots, W_{i_\chi}$, $W_{i_j} \subseteq V_j$,

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$|W_{i_j}| \geqslant \beta |V_j|$, $j = 1, \ldots, \chi$, and for every $h, l \in \{i_1, i_2, \ldots, i_\chi\}$, $l - h \geqslant g - 1$, all but ψ edges $\{u, v\}$ of $G[W_h \cup W_l]$, $u \in W_h$, $v \in W_l$, are $(W_{i_j})_{j=h+1}^{l-1}$-connected.

Definition 4. Let m and $\chi > 1$ be positive integers such that $m \equiv 0 \pmod{\chi}$, say $m = \chi n$. Let $V_i, i = 1, \ldots, \chi$, be pairwise disjoint sets of cardinality n. Let p, possibly a function of m, satisfy $0 \leqslant p \leqslant 1$. The *$\chi$-partite random graph* $\mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ is a probability space on the set of all χ-partite graphs on the vertex set $V = V_1 \cup \cdots \cup V_\chi$ where the sets $V_i, i \in [\chi]$, are independent. The probability measure on $\mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ is given by

$$\mathrm{Prob}(G) = p^{e(G)}(1-p)^{\binom{\chi}{2}n^2 - e(G)}$$

for $G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$.

In Definition 4, we can equivalently think of each edge of the graph chosen independently with probability p.

Definition 5. Let $g \geqslant 3$ be an integer. Let Σ be the set of all graphs on a finite vertex set. We define a mapping $D_g : \Sigma \to \Sigma$ as follows. For $G = (V, E) \in \Sigma$ we set $D_g(G)$ to be the graph obtained from G by deleting all edges which belong to cycles of length at most $g - 1$, *i.e.*, formally, $D_g(G) = (V, E_g)$ where $E_g = E \setminus \{e \in E \ ; \ e \text{ belongs to a cycle of length at most } g - 1 \text{ in } G\}$.

To shorten our notation, we let $e_g(G) = e(D_g(G))$ stand for the number of edges of $D_g(G)$.

Proposition 5.1. *Let χ and g be integers such that $\chi \geqslant g \geqslant 3$. Let $0 < \beta < 1$, and $0 < \epsilon < 1$, and $0 < \lambda < \frac{1}{(g-1)(3g-5)}$ be real numbers. Then, for every sufficiently large integer $m, m \equiv 0 \pmod{\chi}$, there is a graph H satisfying the following properties.*

(a) *H is a χ-partite graph on m vertices with $V(H) = V_1 \cup \cdots \cup V_\chi$ where the sets V_i are independent and of the same cardinality $n, i \in [\chi]$.*

(b) *$\chi(H) = \chi$, $\mathrm{girth}(H) \geqslant g$.*

(c) *For all indices $i, j \in [\chi], i < j$, and for all $B_i \subseteq V_i$, $B_j \subseteq V_j$ such that $|B_i| \geqslant \beta |V_i|$, $|B_j| \geqslant \beta |V_j|$ the following holds:*

$$|e(B_i, B_j) - |B_i||B_j| n^{-1+\frac{1}{g-1}+\lambda}| \leqslant \epsilon |B_i||B_j| n^{-1+\frac{1}{g-1}+\lambda}.$$

(d) $|e(H) - \binom{\chi}{2} n^{1+\frac{1}{g-1}+\lambda}| \leqslant \epsilon \binom{\chi}{2} n^{1+\frac{1}{g-1}+\lambda}$.

(e) *The graph H has the $(\beta, \frac{1}{\ln n} \cdot \beta^2 n^{1+\frac{1}{g-1}+\lambda})$-property.*

Proof. Fix χ, g, β, ϵ, and λ satisfying the assumptions of the proposition. Let $m \equiv 0 \pmod{\chi}$ be a sufficiently large integer and set $n = m/\chi$.

For the rest of our paper we fix the following two parameters:

$$p = n^{-1+\frac{1}{g-1}+\lambda} \tag{5.1}$$

and

$$\psi = \frac{1}{\ln n} \cdot \beta^2 n^2 p. \tag{5.2}$$

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We are going to show the existence of a graph H with properties (a)–(e) via the probabilistic method. In order to do that we define the following events on $\mathscr{G}^{\mathrm{PART}}(n,\chi;p)$:

$$\begin{aligned}
\mathscr{A} &= \{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : \chi(D_g(G)) = \chi\},\\
\mathscr{B}(\alpha,\epsilon) &= \{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : \forall\, i,j \in \{1,\dots,\chi\}, i<j, \forall\, B_i \subseteq V_i, B_j \subseteq V_j,\\
&\qquad |B_i|,|B_j| \geqslant \alpha n : (1-\epsilon)|B_i||B_j|p \leqslant e_g(B_i,B_j) \leqslant (1+\epsilon)|B_i||B_j|p\},\\
&\qquad \text{where } 0<\alpha\leqslant 1,\\
\mathscr{C}(\beta,\gamma) &= \{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : D_g(G) \text{ has the } (\beta,\gamma)\text{-property}\}.
\end{aligned}$$

The proof relies on the following three lemmas. We prove Lemmas 5.2 and 5.3 later in this section. The proof of Lemma 5.4 is postponed to Section 6.

Lemma 5.2. $\mathrm{Prob}(\mathscr{A}) = 1 - o(1)$ *as* $n \to \infty$.

Lemma 5.3. *For every* $\alpha, 0<\alpha\leqslant 1$, $\mathrm{Prob}(\mathscr{B}(\alpha,\epsilon)) = 1-o(1)$ *as* $n\to\infty$.

Lemma 5.4. $\mathrm{Prob}(\mathscr{C}(\beta,\psi)) = 1-o(1)$, *as* $n\to\infty$.

In order to show that the property in (d) holds as well, we set $\mathscr{D} = \{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : (1-\epsilon)\binom{\chi}{2}n^2p \leqslant e(D_g(G)) \leqslant (1+\epsilon)\binom{\chi}{2}n^2p\}$.

Set $\alpha = 1$ in Lemma 5.3. We get

$$\begin{aligned}
\mathrm{Prob}\big(\{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : (1-\epsilon)n^2p \leqslant e_g(V_i,V_j) \leqslant (1+\epsilon)n^2p \ \text{ for all } i<j\}\big)\\
= 1-o(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{Prob}(\mathscr{D}) &= \mathrm{Prob}\left(\left\{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : (1-\epsilon)\binom{\chi}{2}n^2p \leqslant e(D_g(G))\right.\right.\\
&= \left.\left.\sum_{1\leqslant i<j\leqslant\chi} e_g(V_i,V_j) \leqslant (1+\epsilon)\binom{\chi}{2}n^2p\right\}\right) \qquad (5.3)\\
&= 1-o(1).
\end{aligned}$$

Lemmas 5.2–5.4 and (5.3) immediately imply that

$$\mathrm{Prob}\big(\mathscr{A}\cap\mathscr{B}(\beta,\epsilon)\cap\mathscr{C}(\beta,\psi)\cap\mathscr{D}\big) = 1-o(1). \qquad (5.4)$$

Thus there is a graph $G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p)$ such that $H = D_g(G)$ satisfies properties (a)–(e) in the statement of the proposition. Note that the definition of $D_g(G)$ implies that $\mathrm{girth}(D_g(G)) \geqslant g$. □

We are going to use the following fact and claim in the proofs of Lemmas 5.2 and 5.3. We maintain notation introduced in the proof of Proposition 5.1.

A random variable $\mathbf{Y}$ is called an *indicator random variable* if it takes only values 0 and 1. The following fact is proved in [1] (*cf.* Corollary A.14, p. 239).

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Fact 5.5. *Let* $\mathbf{Y}$ *be the sum of mutually independent indicator random variables. For all* $\delta > 0$,

$$\mathrm{Prob}\big(|\mathbf{Y} - \mathbf{E}[\mathbf{Y}]| > \delta \mathbf{E}[\mathbf{Y}]\big) < 2\exp\big(-ch(\delta)\mathbf{E}[\mathbf{Y}]\big)$$

where $ch(\delta) > 0$ *depends only on* δ.

Claim 5.6. *Let* $0 < v < 1$ *and* $0 < \alpha \leqslant 1$ *be real numbers and let* i, j, $1 \leqslant i < j \leqslant \chi$, *be integers. Set*

$$\mathscr{U}\mathrm{nif}_{i,j}(\alpha, v) = \{G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p) : \forall B_i \subseteq V_i, B_j \subseteq V_j, |B_i| \geqslant \alpha n, |B_j| \geqslant \alpha n,$$
$$e_g(B_i, B_j) > (1 - v)|B_i||B_j|p\}.$$

Then for n *large enough*

$$\mathrm{Prob}\big(\cap_{1 \leqslant i < j \leqslant \chi} \mathscr{U}\mathrm{nif}_{i,j}(\alpha, v)\big) \geqslant 1 - \frac{\chi^2 g}{\ln n}.$$

Proof. Recall that our probability $p = n^{-1+\frac{1}{g-1}+\lambda}$ where $0 < \lambda < \frac{1}{(g-1)(3g-5)}$.

Let $\mathbf{XC}_l = \mathbf{XC}_l(G)$ be a random variable counting the l-cycles in $G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ and $\mathscr{X}_l$ be the event that $\mathbf{XC}_l > (\chi np)^l \cdot \ln n$. We claim that for every $l \geqslant 3$

$$\mathrm{Prob}(\mathscr{X}_l) \leqslant \frac{1}{\ln n}. \tag{5.5}$$

The expected number of l-cycles in $\mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ is bounded from above by $(\chi np)^l$ and, consequently, we infer (5.5) via the Markov inequality. Inequality (5.5) implies that

$$\mathrm{Prob}\big(\cup_{l=3}^{g-1} \mathscr{X}_l\big) < \frac{g}{\ln n}.$$

Hence,

$$\mathrm{Prob}\big(\cap_{l=3}^{g-1} \bar{\mathscr{X}}_l\big) > 1 - \frac{g}{\ln n}, \tag{5.6}$$

where $\bar{\mathscr{X}}_l$ denotes the complementary event to $\mathscr{X}_l$. Therefore, the following holds with probability greater than $1 - g/\ln n$:

$$\text{for every } l, 3 \leqslant l \leqslant g-1 : \mathbf{XC}_l \leqslant (\chi n^{\frac{1}{g-1}+\lambda})^l \cdot \ln n \leqslant \chi^{g-1} n^{1+\lambda(g-1)} \cdot \ln n. \tag{5.7}$$

This implies that the total number of edges of $\mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ in cycles of length at most $g - 1$ is bounded from above by

$$\sum_{l=3}^{g-1} l \cdot \mathbf{XC}_l \leqslant \sum_{l=3}^{g-1} l \cdot \chi^{g-1} n^{1+\lambda(g-1)} \cdot \ln n \leqslant g^2 \chi^{g-1} n^{1+\lambda(g-1)} \cdot \ln n \tag{5.8}$$

with probability greater than $1 - g/\ln n$.

Fix $i, j, 1 \leqslant i < j \leqslant \chi$. Let B_i, B_j be fixed sets $B_i \subseteq V_i$, $B_j \subseteq V_j$, $|B_i| \geqslant \alpha n$, $|B_j| \geqslant \alpha n$, and set $\mathbf{X}_{i,j} = e(B_i, B_j)$ to be the number of edges between B_i and B_j in a graph $G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$. The random variable $\mathbf{X}_{i,j}$ possesses the binomial distribution $\mathrm{Bi}(|B_i||B_j|, p)$,

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and thus the probability that $e(B_i, B_j) < (1 - \frac{v}{2})|B_i||B_j|p$ is exponentially small. Indeed, as

$$\sum_{k<(1-\frac{v}{2})|B_i||B_j|p} \binom{|B_i||B_j|}{k}(1-p)^{|B_i||B_j|-k}p^k = \mathrm{Prob}\left(\mathbf{X}_{i,j} < \left(1-\frac{v}{2}\right)|B_i||B_j|p\right),$$

Fact 5.5 implies

$$\begin{aligned}\mathrm{Prob}\left(\mathbf{X}_{i,j} < \left(1-\frac{v}{2}\right)|B_i||B_j|p\right) &\leqslant 2\exp(-c'|B_i||B_j|p)\\ &\leqslant \exp(-c\alpha^2 n^2 p)\\ &= \exp(-c\alpha^2 n^{1+\frac{1}{g-1}+\lambda}),\end{aligned}$$

where $c' = ch(v/2)$. Thus, for n large enough the probability that, for every choice of $B_i \subseteq V_i$, $B_j \subseteq V_j$, $|B_i| \geqslant \alpha n$, $|B_j| \geqslant \alpha n$, the number of edges $e(B_i, B_j) < (1 - \frac{v}{2})|B_i||B_j|p$ is at most $\binom{n}{\alpha n}^2 \cdot e^{-c\alpha^2 n^{1+\frac{1}{g-1}+\lambda}} < e^{-\tilde{c}\alpha^2 n^{1+\frac{1}{g-1}+\lambda}}$ for a suitable constant $\tilde{c} > 0$. Hence, the number of edges between any two sets $B_i \subseteq V_i$ and $B_j \subseteq V_j$, $|B_i| \geqslant \alpha n$, $|B_j| \geqslant \alpha n$, in $\mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ satisfies asymptotically almost surely

$$e(B_i, B_j) \geqslant \left(1-\frac{v}{2}\right)\alpha^2 n^2 p = \left(1-\frac{v}{2}\right)\alpha^2 n^{1+\frac{1}{g-1}+\lambda}. \tag{5.9}$$

Recall that $e_g(G)$ refers to the number of edges in the graph $D_g(G)$. We wish to show that $e_g(B_i, B_j)$ is almost the same as $e(B_i, B_j)$. Indeed, as $\frac{v}{2}\alpha^2 n^2 p \geqslant g^2\chi^{g-1}n^{1+\lambda(g-1)} \cdot \ln n$ for n sufficiently large, we have for any B_i, B_j as above that

$$\begin{aligned}e_g(B_i, B_j) &\geqslant e(B_i, B_j) - \sum_{l=3}^{g-1} l \cdot \mathbf{XC}_l\\ &\geqslant \left(1-\frac{v}{2}\right)|B_i||B_j|p - g^2\chi^{g-1}n^{1+\lambda(g-1)} \cdot \ln n\\ &\geqslant \left(1-\frac{v}{2}\right)|B_i||B_j|p - \frac{v}{2}\alpha^2 n^2 p\\ &\geqslant (1-v)|B_i||B_j|p\end{aligned}$$

holds with probability greater than $1 - 2g/\ln n - e^{-\tilde{c}\alpha^2 n^{1+\frac{1}{g-1}+\lambda}}$. In other words,

$$\mathrm{Prob}(\mathscr{U}\mathrm{nif}_{i,j}(\alpha, v)) \geqslant 1 - 2g/\ln n - e^{-\tilde{c}\alpha^2 n^{1+\frac{1}{g-1}+\lambda}}.$$

This implies that

$$\mathrm{Prob}\left(\cap_{1\leqslant i<j\leqslant\chi}\mathscr{U}\mathrm{nif}_{i,j}(\alpha, v)\right) \geqslant 1 - 2\binom{\chi}{2}\frac{g}{\ln n} - \binom{\chi}{2}e^{-\tilde{c}\alpha^2 n^{1+\frac{1}{g-1}+\lambda}} \geqslant 1 - \frac{\chi^2 g}{\ln n}. \qquad \square$$

Proof of Lemma 5.2. Note that for every $G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ the chromatic number of $D_g(G)$ satisfies $\chi(D_g(G)) \leqslant \chi$.

Let $G \in \cap_{1\leqslant i<j\leqslant\chi}\mathscr{U}\mathrm{nif}_{i,j}(\frac{1}{\chi-1}, \frac{1}{2})$, where $\mathscr{U}\mathrm{nif}_{i,j}(\frac{1}{\chi-1}, \frac{1}{2})$ are events defined in Claim 5.6.We claim that $\chi(D_g(G)) = \chi$. Assume on the contrary, that $\chi(D_g(G)) < \chi$. Let φ be a proper

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$(\chi-1)$-colouring of $D_g(G)$. For $i \in [\chi]$, we set $B_i \subset V_i$ to be the set of vertices coloured with the most frequent colour which the colouring φ assigns to the vertices of V_i. Thus,

$$|B_i| \geqslant \frac{1}{\chi-1}|V_i| = \frac{1}{\chi-1}n.$$

As φ is using only $\chi-1$ colours, there are at least two sets B_i, B_j, $i, j \in [\chi]$, $i \neq j$, coloured by the same colour under φ. However, as $G \in \cap_{1\leqslant i<j\leqslant\chi} \mathscr{U}\mathrm{nif}_{i,j}(\frac{1}{\chi-1}, \frac{1}{2})$ there is an edge between the sets B_i and B_j in the graph $D_g(G)$ (*cf.* Claim 5.6). This is a contradiction.

Hence, $\mathrm{Prob}(\mathscr{A}) = \mathrm{Prob}(\{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : \chi(D_g(G)) = \chi\}) = 1 - o(1)$. □

Proof of Lemma 5.3. Let $0 < \alpha \leqslant 1$ and $0 < \epsilon < 1$ be real numbers. Let $\mathscr{U}\mathrm{nif}_{i,j}(\alpha,\epsilon)$ be the events defined in Claim 5.6. Thus, by Claim 5.6,

$$\begin{aligned}
&\mathrm{Prob}\big(\cap_{1\leqslant i<j\leqslant\chi} \mathscr{U}\mathrm{nif}_{i,j}(\alpha,\epsilon)\big)\\
&\quad= \mathrm{Prob}\big(\{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : \forall\, i,j \in \{1,\dots,\chi\}, i<j,\\
&\qquad\qquad \forall\, B_i \subseteq V_i, B_j \subseteq V_j, |B_i|,|B_j| \geqslant \alpha n, e_g(B_i,B_j) > (1-\epsilon)|B_i||B_j|p\}\big)\\
&\quad= 1 - o(1). \qquad (5.10)
\end{aligned}$$

Note that for any two subsets of vertices A, B of $G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p)$ one has $e_g(A,B) \leqslant e(A,B)$. Let $i, j \in [\chi]$, $i < j$. We are going to show that for fixed $B_i \subseteq V_i$, $B_j \subseteq V_j$, $|B_i|, |B_j| \geqslant \alpha n$, the probability of $e(B_i,B_j) > (1+\epsilon)|B_i||B_j|p$ is exponentially small. Let $\mathbf{X}_{i,j} = e(B_i,B_j)$ be a random variable. Note that $\mathbf{X}_{i,j}$ has the binomial distribution $\mathrm{Bi}(|B_i||B_j|,p)$. Using Fact 5.5 we get

$$\begin{aligned}
\mathrm{Prob}\big(\mathbf{X}_{i,j} > (1+\epsilon)|B_i||B_j|p\big) &< 2\exp(-c'|B_i||B_j|p)\\
&\leqslant \exp(-c\alpha^2 n^2 p)\\
&= \exp(-c\alpha^2 n^{1+\frac{1}{g-1}+\lambda}),
\end{aligned}$$

where $c' = ch(\epsilon)$. Thus, the probability that for every choice of $B_i \subseteq V_i$, $B_j \subseteq V_j$, $|B_i| \geqslant \alpha n$, $|B_j| \geqslant \alpha n$ the number of edges $e(B_i,B_j) > (1+\epsilon)|B_i||B_j|p$ is $o(1)$ as $n \to \infty$. Hence, for $i, j \in [\chi]$, $i < j$, $\mathrm{Prob}\big(\{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : \forall\, B_i \subseteq V_i, B_j \subseteq V_j, |B_i|,|B_j| \geqslant \alpha n, e(B_i,B_j) \leqslant (1+\epsilon)|B_i||B_j|p\}\big) = 1 - o(1)$, and thus

$$\begin{aligned}
&\mathrm{Prob}\big(\{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p) : \forall i,j \in [\chi], i<j, \forall B_i \subseteq V_i, B_j \subseteq V_j,\\
&\qquad |B_i|,|B_j| \geqslant \alpha n, e(B_i,B_j) \leqslant (1+\epsilon)|B_i||B_j|p\}\big)\\
&= 1 - o(1). \qquad (5.11)
\end{aligned}$$

Hence, using (5.10) and (5.11) we get $\mathrm{Prob}(\mathscr{B}(\alpha,\epsilon)) = 1 - o(1)$. □

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6. The (β, ψ)-property for the random graph $\mathscr{G}^{\mathrm{PART}}(n, \chi; p)$: Proof of Lemma 5.4

In this section we give a proof of Lemma 5.4. Since the proof is quite technical, we postpone the main part of the proof of Lemma 5.4 to Section 7 where we establish that for every $W_1, \ldots, W_g$, $W_j \subseteq V_{i_j}$, $|W_j| \geqslant \beta n$, $j \in [g]$ all but $\psi/2$ edges of $G[W_1 \cup W_g]$, $G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ (*cf.* (5.2) for the definition of ψ) are $(W_j)_{j=2}^{g-1}$-connected with probability of failure exponentially small in $n \log n$. This establishes the main part of the proof of Lemma 5.4.

For *fixed* $W_1, \ldots, W_g$ each being a subset of a different set of vertices V_i of $\mathscr{G}^{\mathrm{PART}}(n, \chi; p)$ and $|W_j| \geqslant \beta n$, $j = 1, \ldots, g$, let $\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}) = \{G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p) : \text{all but } \frac{1}{2}\psi \text{ edges } \{u, v\} \text{ of } G[W_1 \cup W_g], u \in W_1, v \in W_g, \text{ are } (W_j)_{j=2}^{g-1}\text{-connected}\}$ (*cf.* Definition 3). Also, let

$$\mathscr{C}\mathrm{onn}\left(\beta, \frac{\psi}{2}\right) = \bigcap_{W_1, \ldots, W_g} \mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}\right),$$

where the intersection runs over all sets $W_1, \ldots, W_g$, each being a subset of a different set of vertices V_i and $|W_j| \geqslant \beta n, j = 1, \ldots, g$. In other words $\mathscr{C}\mathrm{onn}(\beta, \frac{\psi}{2})$ is the event that $\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2})$ holds for all choices of $(W_j)_{j=1}^{g}$, each W_j belonging to a different set of vertices V_i, and $|W_j| \geqslant \beta n, j \in [g]$. Finally, let us denote by $\neg\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2})$ the event complementary to $\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2})$.

The rest of this section is devoted to the proof of Lemma 5.4, *i.e.*, to the proof that the event $\mathscr{C}(\beta, \psi)$ holds with probability $1 - o(1)$. This will be done in three steps.

(a) First, we show that

$$\mathrm{Prob}\left(\mathscr{C}\mathrm{onn}\left(\beta, \frac{\psi}{2}\right)\right) = 1 - o(1) \text{ as } n \to \infty. \tag{6.1}$$

(b) Second, we are going to show that when deleting edges in short cycles, *i.e.*, edges in cycles of length less than g, we get that with probability $1 - o(1)$ the following holds: no matter how the sets $W_1, \ldots, W_g$ are chosen after this deletion at most $\frac{1}{2}\psi$ edges become $(W_j)_{j=2}^{g-1}$-disconnected.

(c) Third, combining (a) and (b) we will infer that $\mathrm{Prob}\big(\mathscr{C}(\beta, \psi)\big) = 1 - o(1)$.

(a) In order to prove (6.1) it will be sufficient to prove the following lemma. We defer its proof to Section 7.

Lemma 6.1. *For any choice of the sets $W_1, \ldots, W_g$, as above*

$$\mathrm{Prob}\left(\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}\right)\right) < \exp(-cn \log n)$$

where c is a positive constant.

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Indeed, we have

$$\neg\mathscr{C}\mathrm{onn}\left(\beta,\frac{\psi}{2}\right)=\left\{G\in\mathscr{G}^{\mathrm{PART}}(n,\chi;p):\exists\ W_1,\dots,W_g \text{ so that } \neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)\right\}$$
$$\subseteq\bigcup_{W_1,\dots,W_g}\left\{G\in\mathscr{G}^{\mathrm{PART}}(n,\chi;p):\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)\text{ holds}\right\}$$

where the union runs over all sets $W_1,\dots,W_g$, each being a subset of a different set of vertices V_i and $|W_j|\geqslant\beta n, j=1,\dots,g$.

Hence, by Lemma 6.1,

$$\begin{aligned}\mathrm{Prob}\left(\neg\mathscr{C}\mathrm{onn}\left(\beta,\frac{\psi}{2}\right)\right)&=\mathrm{Prob}\left(\left\{G\in\mathscr{G}^{\mathrm{PART}}(n,\chi;p):\exists\ W_1,\dots,W_g\right.\right.\\&\qquad\left.\left.\text{so that }\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)\right\}\right)\\&\leqslant\sum_{W_1,\dots,W_g}\mathrm{Prob}\left(\left\{G\in\mathscr{G}^{\mathrm{PART}}(n,\chi;p):\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)\right\}\right)\\&\leqslant\chi^g\left[\sum_{l=\lceil\beta n\rceil}^{n}\binom{n}{l}\right]^g\cdot\exp(-cn\log n)\\&\leqslant\chi^g\cdot 2^{gn}\cdot\exp(-cn\log n)\\&=o(1).\end{aligned}$$

Thus the property $\mathscr{C}\mathrm{onn}(\beta,\frac{\psi}{2})$ holds asymptotically almost surely for $\mathscr{G}^{\mathrm{PART}}(n,\chi;p)$.

(b) For a graph G and for fixed $W_1,\dots,W_g$ let $\mathbf{Cyc}_g(G,(W_j)_{j=2}^{g-1})$ be a random variable counting the g-cycles $v_1,\dots,v_g,v_j\in W_j$, such that at least one of the edges $\{v_1,v_g\}$ and $\{v_j,v_{j+1}\}$, $j=1,\dots,g-1$, is contained in a 'short cycle' of G, *i.e.*, in a cycle of length at most $g-1$. Note that the vertex set of the short cycle does not necessary belong to the sets W_j, $j=1,\dots,g$.

Set

$$\mathscr{S}\mathrm{hort}_g\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)=\left\{G\in\mathscr{G}^{\mathrm{PART}}(n,\chi;p):\mathbf{Cyc}_g\left(G,(W_j)_{j=2}^{g-1}\right)\leqslant\frac{\psi}{2}\right\}$$

and

$$\mathscr{S}\mathrm{hort}_g\left(\beta,\frac{\psi}{2}\right)=\bigcap_{W_1,\dots,W_g}\mathscr{S}\mathrm{hort}_g\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)$$

where the intersection is taken over all sets $W_1,\dots,W_g$ each being a subset of a different set of vertices V_i and $|W_j|\geqslant\beta n, j=1,\dots,g$.

Claim 6.2.

$$\mathrm{Prob}\left(\mathscr{S}\mathrm{hort}_g\left(\beta,\frac{\psi}{2}\right)\right)=1-o(1).\tag{6.2}$$

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Proof. In order to show (6.2), let $\mathbf{X} = \mathbf{X}(G), G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$, be a random variable counting pairs of cycles C_1, C_2 such that the length of the cycle C_1 is g, the length of the cycle C_2 is at most $g-1$, and C_1 and C_2 share at least one edge.

To prove (6.2) it is enough to show that $\mathrm{Prob}(\mathbf{X}(G) \geqslant \frac{1}{2}\psi) = o(1)$. We have

$$\mathbf{E}[\mathbf{X}] = O(n^{2g-3}p^{2g-2}).$$

Hence, the Markov inequality together with the fact that $\mathbf{E}[\mathbf{X}] = O(n^{2g-3}p^{2g-2}) = o(\frac{1}{2}\psi)$ imply $\mathrm{Prob}(\mathbf{X} \geqslant \frac{1}{2}\psi) \leqslant \frac{\mathbf{E}[\mathbf{X}]}{\psi/2} = o(1)$. □

(c) The proof of Lemma 5.4 follows now from the next claim.

Claim 6.3.

$$\mathscr{S}\mathrm{hort}_g\left(\beta, \frac{\psi}{2}\right) \cap \mathscr{C}\mathrm{onn}\left(\beta, \frac{\psi}{2}\right) \subseteq \mathscr{C}(\beta, \psi). \tag{6.3}$$

Proof. Let $G \in \mathscr{S}\mathrm{hort}_g(\beta, \frac{\psi}{2}) \cap \mathscr{C}\mathrm{onn}(\beta, \frac{\psi}{2})$ and let sets $W_1, \ldots, W_g$ each being a subset of a different set of vertices V_i and $|W_j| \geqslant \beta n$, $j = 1, \ldots, g$, be given. Since $G \in \mathscr{C}\mathrm{onn}(\beta, \frac{\psi}{2})$, $G[W_1 \cup \cdots \cup W_g]$ has at most $\frac{1}{2}\psi$ $(W_j)_{j=2}^{g-1}$-disconnected edges. On the other hand, as $G \in \mathscr{S}\mathrm{hort}_g(\beta, \frac{\psi}{2})$ the number of g-cycles $(v_1, \ldots, v_g)$, $v_i \in W_i$, $i = 1, \ldots, g$, with an edge contained in a short cycle is at most $\frac{\psi}{2}$ and, consequently, the number of edges of $G[W_1 \cup \cdots \cup W_g]$ that become $(W_j)_{j=2}^{g-1}$-disconnected after deletion of cycles of length at most $g-1$ does not exceed $\frac{\psi}{2}$. □

This completes the proof of Lemma 5.4. □

7. The property $\mathscr{C}$onn: Proof of Lemma 6.1

In this section we prove Lemma 6.1. We need to show that for fixed sets $W_1, \ldots, W_g$ each set being a subset of a different set V_i and $|W_j| \geqslant \beta n, j = 1, \ldots, g$, the probability $\mathrm{Prob}\big(G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$: the number of $(W_j)_{j=2}^{g-1}$-disconnected edges is bigger than $\frac{\psi}{2}\big)$ $< \exp(-cn \log n)$.

Let us recall the Janson inequality. Let $(I_\omega)_{\omega \in \Omega}$ be a collection of independent 0, 1-valued random variables and let $J_\gamma, \gamma \in \Gamma$, be a family of subsets of the index set Ω. Let $T_\gamma = \prod_{\omega \in J_\gamma} I_\omega$ and $T = \sum_{\gamma \in \Gamma} T_\gamma$. Further, set

$$\Delta = \sum_{J_{\gamma_1} \cap J_{\gamma_2} \neq \emptyset} \mathbf{E}[T_{\gamma_1} T_{\gamma_2}]$$

and

$$v = \max_{\gamma \in \Gamma} \mathrm{Prob}(T_\gamma = 1).$$

Fact 7.1. (Janson Inequality (see [1], p. 96)). *With the notation as above,*

$$\mathrm{Prob}(T = 0) \leqslant \exp\left(\frac{1}{1-v} \cdot \frac{\Delta}{2}\right) \prod_{\gamma \in \Gamma} \mathrm{Prob}(T_\gamma = 0). \tag{7.1}$$

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In other words, let $\mathscr{S}_\gamma$ be the event that $T_\gamma = 1$, *i.e.*, all $I_\omega, \omega \in J_\gamma$, equal 1. Then (7.1) can be rewritten as

$$\mathrm{Prob}\big(\wedge_{\gamma\in\Gamma}\neg\mathscr{S}_\gamma\big) \leqslant \exp\left(\frac{1}{1-\nu}\cdot\frac{\Delta}{2}\right)\prod_{\gamma\in\Gamma}\mathrm{Prob}(\neg\mathscr{S}_\gamma).$$

In order to proceed with the proof of Lemma 6.1, let us fix sets $W_1,\dots,W_g$, each of cardinality at least βn as above. Set

$$t = \lceil(\beta n)^{1-\lambda(g-1)/2}\rceil. \tag{7.2}$$

For two vertices $x \in W_1$ and $y \in W_g$, we say that there is a $(W_j)_{j=2}^{g-1}$-path from x to y if there are vertices $w_1 = x, w_2,\dots,w_g = y$ such that $w_i \in W_i$, $i = 1,\dots,g$, and all the sets $\{w_i, w_{i+1}\}$, $i = 1,\dots,g-1$, are edges in the graph $G[W_1 \cup \cdots \cup W_g]$. Let $\mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1}, t) = \{G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p)$: there are $2t$ vertices $x_1,\dots,x_t \in W_1$ and $y_1,\dots,y_t \in W_g$ such that there is no $(W_j)_{j=2}^{g-1}$-path x_i to y_i for all $i = 1,\dots,t\}$.

We are going to show that the probability $\mathrm{Prob}(\neg\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2})$ is exponentially small in $n\log n$ (*cf.* Lemma 6.1). We have

$$\begin{aligned}\mathrm{Prob}\left(\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)\right) &\leqslant \mathrm{Prob}\big(\mathscr{D}\mathrm{isc}\big((W_j)_{j=2}^{g-1},t\big)\big) \\ &+ \mathrm{Prob}\left(\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}\right)\ \wedge\ \neg\mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1},t)\right).\end{aligned} \tag{7.3}$$

Note that the property $\neg\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}) \wedge \neg\mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1})$ implies that there are two sets $A \subset W_1$ and $B \subset W_g$, $|A| = |B| < t$, and there are at least $\psi/2$ $(W_j)_{j=2}^{g-1}$-disconnected edges such that all of them are incident to A and/or B. In Claims 7.2 and 7.3 we are going to show that both probabilities $\mathrm{Prob}(\mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1},t))$ and $\mathrm{Prob}(\neg\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1},\beta,\frac{\psi}{2}) \wedge \neg\mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1},t))$ are asymptotically much smaller than $\exp(-n\log n)$.

Claim 7.2. $\mathrm{Prob}(\mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1},t)) < \exp(-cn^{1+\lambda(g-1)/2})$ *where c is a positive constant.*

Proof. Recall $t = \lceil(\beta n)^{1-\lambda(g-1)/2}\rceil$. Let us fix a pair of vertices $x_i, y_i, i \in [t]$. For each $i = 1,2,\dots,t$ let A^i be the set of all possible $(W_j)_{j=2}^{g-1}$-paths joining x_i with y_i. Clearly $\alpha := |A^i| = \prod_{j=2}^{g-1}|W_j|$. For each $a \in A^i$, *i.e.*, for each $(g-2)$-tuple of vertices $a = (w_2,\dots,w_{g-1})$, let $\mathscr{S}_{(i,a)}$ be the event that $x_i w_2 \dots w_{g-1} y_i$ is a $(W_j)_{j=2}^{g-1}$-path in $G \in \mathscr{G}^{\mathrm{PART}}(n,\chi;p)$. The event $\wedge_{(i,a)}\neg\mathscr{S}_{(i,a)}$ means that there are no $(W_j)_{j=2}^{g-1}$-paths for any $x_i, y_i, i \in [t]$.

We have

$$\mathrm{Prob}(\mathscr{S}_{(i,a)}) = p^{g-1} = n^{-g+2+\lambda(g-1)}$$

and

$$\begin{aligned}\prod_{(i,a)}\mathrm{Prob}(\neg\mathscr{S}_{(i,a)}) &= (1-p^{g-1})^{\alpha\cdot t} \\ &\leqslant \exp(-p^{g-1}\alpha\cdot t) \\ &\leqslant \exp(-\beta^{g-2}n^{\lambda(g-1)}t).\end{aligned}$$

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Using the notation introduced above we write $(i,a) \sim (j,b)$ if and only if the paths involved in the events $\mathscr{S}_{(i,a)}$ and $\mathscr{S}_{(j,b)}$ share at least one edge. More precisely, for $k = 1,\dots,g-1$, and $i,j = 1,\dots,t$, and $a = (w_2,\dots,w_{g-1})$, and $b = (w'_2,\dots,w'_{g-1})$ we write $(i,a) \sim_k (j,b)$ if and only if the paths $P_{i,a} := x_i w_2 \dots w_{g-1} y_i$ and $P_{j,b} := x_j w'_2 \dots w'_{g-1} y_j$ share precisely k edges.

If $(i,a) \sim_k (j,b)$,

$$\mathrm{Prob}(\mathscr{S}_{(i,a)} \wedge \mathscr{S}_{(j,b)}) = p^{2(g-1)-k}.$$

To give a bound on

$$\Delta = \sum_{(i,a)\sim(j,b)} \mathrm{Prob}(\mathscr{S}_{(i,a)} \wedge \mathscr{S}_{(j,b)}) = \sum_{k=1}^{g-3} \sum_{(i,a)\sim_k(j,b)} \mathrm{Prob}(\mathscr{S}_{(i,a)} \wedge \mathscr{S}_{(j,b)})$$

we distinguish two cases $i \neq j$ and $i = j$ and denote the corresponding contributions to Δ by Δ_1 and Δ_2, respectively.

If $i \neq j$, note that if $P_{(i,a)}$ and $P_{(j,b)}$ share k edges they must have $s \geqslant k+1$ vertices in common. Observe that the number of possible pairs $(P_{(i,a)}, P_{(j,b)})$ is bounded from above by

$$\sum_{(i,a)\sim_k(j,b), i\neq j} 1 \leqslant \sum_{s=k+1}^{g-2} \binom{t}{2} n^{2(g-2)-s} \binom{g-2}{k} = O(t^2 n^{2(g-2)-k-1}),$$

and hence

$$\Delta_1 = \sum_{k=1}^{g-3} O\left(p^{2(g-1)-k} t^2 n^{2(g-2)-k-1}\right).$$

To estimate Δ_2 we need a bit more careful analysis. Observe that in the case the common part of $P_{(i,a)} \cap P_{(i,b)}$ has s vertices, k edges, and h components, then $s = k + h$. Since $P_{(i,a)}$ and $P_{(i,b)}$ share initial and final vertices, we must have $h \geqslant 2$. Hence

$$|V(P_{(i,a)}) \cap V(P_{(i,b)})| \geqslant k+2,$$

and thus

$$|V(P_{(i,a)}) \cap V(P_{(i,b)}) \cap (W_2 \cup \dots \cup W_{g-1})| \geqslant k.$$

Therefore, the number of possible pairs $(P_{(i,a)}, P_{(i,b)})$ is bounded from above by

$$\sum_{(i,a)\sim_k(i,b)} 1 = O\left(\sum_{s=k}^{g-2} t n^{2(g-2)-s}\right) = O(t n^{2(g-2)-k}),$$

and thus

$$\begin{aligned}\Delta_2 &= \sum_{k=1}^{g-3} \sum_{(i,a)\sim_k(i,b)} \mathrm{Prob}(\mathscr{S}_{(i,a)} \wedge \mathscr{S}_{(i,b)}) \\ &= \sum_{k=1}^{g-3} O(p^{2(g-1)-k} t n^{2(g-2)-k}).\end{aligned}$$

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Continuing to bound Δ from above we get

$$\begin{aligned}\Delta &= \Delta_1 + \Delta_2 \\ &= \sum_{k=1}^{g-3} p^{2(g-1)-k} \cdot O\big(t^2 n^{2(g-2)-k-1} + t n^{2(g-2)-k}\big) \\ &= \sum_{k=1}^{g-3} O\big(t n^{2(g-2)-k} p^{2(g-1)-k}\big) \\ &= \sum_{k=1}^{g-3} O\big(t n^{2g-4} p^{2(g-1)} (np)^{-k}\big) \\ &= O\big(n^{1-\frac{1}{g-1}+\lambda(2g-3)}\big).\end{aligned}$$

Applying the Janson inequality for the events $\mathscr{S}_{(i,a)}$, we have

$$\begin{aligned}\mathrm{Prob}\big(\wedge_{i,a} \neg \mathscr{S}_{(i,a)}\big) &\leqslant e^{-\beta^{g-2} n^{\lambda(g-1)} t} \cdot e^{(1+o(1))O(n^{1-\frac{1}{g-1}+\lambda(2g-3)})} \\ &\leqslant \exp\big(-\beta^{g-2} n^{1+\lambda(g-1)/2} + c_1 n^{1-\frac{1}{g-1}+\lambda(2g-3)}\big) \\ &\leqslant \exp\big(-c n^{1+\lambda(g-1)/2}\big)\end{aligned}$$

which holds for any $0 < \lambda < \frac{1}{(g-1)(3g-5)}$. □

Claim 7.3.

$$\mathrm{Prob}\left(\neg \mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}\right) \wedge \neg \mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1}, t)\right) < \exp\big(-c n^{1+\frac{1}{g-1}-\lambda\cdot\frac{g-3}{2}}\big),$$

where c is a positive constant.

Proof. Let sets $W_1, \ldots, W_g$, $|W_i| \geqslant \beta n, i = 1, \ldots, g$, be fixed. Recall $t = \lceil (\beta n)^{1-\lambda\cdot\frac{g-1}{2}} \rceil$.

Let $\mathscr{E}((W_j)_{j=2}^{g-1}, t)$ be the event that there are sets of vertices $A \subset W_1$ and $B \subset W_g$, $|A| = |B| < t$, such that the number of edges $e(A,B) + e(A, W_g \setminus B) + e(W_1 \setminus A, B) > \psi/2$ in $G[W_1 \cup W_g]$. We claim that $\neg \mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}) \wedge \neg \mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1}, t)$ implies $\mathscr{E}((W_j)_{j=2}^{g-1}, t)$. Indeed, let $A \subseteq W_1$, $B \subseteq W_g$ be a maximal pair of sets of vertices such that

(i) $|A| = |B|$

(ii) there exists a labelling of elements of A and B, say $A = \{x_1, \ldots, x_a\}$ and $B = \{y_1, \ldots, y_a\}$, such that there is no $(W_j)_{j=2}^{g-1}$-path x_i to y_i, $i = 1, 2, \ldots, a$.

Then $\neg \mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1}, t)$ implies $|A| = |B| < t$. Moreover, because of the maximality of A, B, all $(W_j)_{j=2}^{g-1}$-disconnected edges must be incident to $A \cup B$. Hence, necessarily,

$$e(A,B) + e(A, W_g \setminus B) + e(W_1 \setminus A, B) > \psi/2. \tag{7.4}$$

To bound the left-hand side of (7.4) in the random graph, first let us fix two sets of vertices $A \subset W_1$ and $B \subset W_g$, $|A| = |B| < t$. We let $\mathbf{Y}_{A,B}$ be a random variable counting edges incident to $A \cup B$ in $G[W_1 \cup W_g]$, $G \in \mathscr{G}^{\mathrm{PART}}(n, \chi; p)$, *i.e.*,

$$\mathbf{Y}_{A,B} = e(A,B) + e(A, W_g \setminus B) + e(W_1 \setminus A, B). \tag{7.5}$$

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Set

$$m = |A| \cdot |B| + |A| \cdot |W_g \setminus B| + |W_1 \setminus A| \cdot |B|.$$

We have the following upper bound on m

$$m \leqslant |A|^2 + 2(n - |A|)|A| \leqslant 2nt.$$

The random variable $\mathbf{Y}_{A,B}$ enjoys a binomial distribution $\mathrm{Bi}(m,p)$ and, hence, $\mathbf{E}[\mathbf{Y}_{A,B}] = mp \leqslant 2ntp$.

Let $\mathbf{Y}^*_{A,B} \sim \mathrm{Bi}(2nt,p)$ be a binomial random variable. Since $\mathbf{Y}_{A,B} \sim \mathrm{Bi}(m,p)$, $m \leqslant 2nt$, and $2\mathbf{E}[\mathbf{Y}^*_{A,B}] \leqslant 4ntp < \psi/2$, we get

$$\begin{aligned}\mathrm{Prob}\big(\mathbf{Y}_{A,B} > \psi/2\big) &\leqslant \mathrm{Prob}\big(\mathbf{Y}^*_{A,B} > \psi/2\big)\\ &\leqslant \mathrm{Prob}\big(\mathbf{Y}^*_{A,B} > 4ntp\big)\\ &\leqslant 2 \cdot \exp\big(-c2ntp\big),\end{aligned}$$

where c is a positive constant. The last inequality follows from Fact 5.5. Substituting for t and p, we get

$$\mathrm{Prob}\big(\mathbf{Y}_{A,B} > \psi/2\big) < \exp\big(-c'n^{1+\frac{1}{g-1}-\lambda\frac{g-3}{2}}\big)$$

for some $c' > 0$. We are ready to give an upper bound on $\mathrm{Prob}\big(\neg\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}) \wedge \neg\mathscr{D}\mathrm{isc}((W_j)_{j=2}^{g-1}, t)\big)$ as promised in the claim. Indeed,

$$\begin{aligned}&\mathrm{Prob}\left(\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}\right) \wedge \neg\mathscr{D}\mathrm{isc}\big((W_j)_{j=2}^{g-1}, t\big)\right)\\ &\quad\leqslant \mathrm{Prob}\big(\mathscr{E}\big((W_j)_{j=2}^{g-1}, t\big)\big)\\ &\quad\leqslant \sum_{\substack{|A|=|B|<t\\ A\subseteq W_1, B\subseteq W_g}} \mathrm{Prob}\big(\mathbf{Y}_{A,B} > \psi/2\big)\\ &\quad\leqslant \sum_{i=1}^{t-1} \binom{|W_1|}{i}\binom{|W_g|}{i} \cdot \exp\big(-cn^{1+\frac{1}{g-1}-\lambda\cdot\frac{g-3}{2}}\big)\\ &\quad\leqslant \sum_{i=1}^{t-1} \binom{n}{i}^2 \cdot \exp\big(-cn^{1+\frac{1}{g-1}-\lambda\cdot\frac{g-3}{2}}\big)\\ &\quad< n2^{2n} \cdot \exp\big(-cn^{1+\frac{1}{g-1}-\lambda\cdot\frac{g-3}{2}}\big)\\ &\quad< \exp\big(-c'n^{1+\frac{1}{g-1}-\lambda\cdot\frac{g-3}{2}}\big).\end{aligned}$$

□

Proof of Lemma 6.1. Using (7.3) and Claims 7.2 and 7.3 we can bound the probability $\mathrm{Prob}(\neg\mathscr{C}\mathrm{onn}((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}))$ from above, as follows:

$$\begin{aligned}\mathrm{Prob}\left(\neg\mathscr{C}\mathrm{onn}\left((W_j)_{j=2}^{g-1}, \beta, \frac{\psi}{2}\right)\right) &< \exp\big(-cn^{1+\frac{1}{2}\lambda(g-1)}\big) + \exp\big(-cn^{1+\frac{1}{g-1}-\frac{1}{2}\lambda(g-3)}\big)\\ &< \exp\big(-cn\log n\big).\end{aligned}$$

This proves Lemma 6.1. □

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8. Proof of Proposition 4.1

Proof of Proposition 4.1. Fix χ, g, ϵ satisfying the assumptions and $\lambda, 0 < \lambda < \frac{1}{(g-1)(3g-5)}$. Let G be a graph which exists by Proposition 5.1, where both ϵ and β are set to $\epsilon/(\chi+1)^2$. Clearly, (a) holds. By Proposition 5.1, there is a χ-colouring of G such that the colour classes $V_1, \ldots, V_\chi$ satisfy $|V_i| = n$ for each $i = 1, \ldots, \chi$. We fix such a χ-colouring of G.

Let $\vec{G}$ be an acyclic orientation of G. We define a relation $\prec$ on $V(G) \times V(G)$, by $x \prec y$, $x, y \in V(G)$, if and only if there is a directed path from the vertex x to the vertex y in $\vec{G}$. Let $<$ be a linear ordering which extends the relation $\prec$. Let $x_1, \ldots, x_m$ be the enumeration of all vertices of G in the linear ordering, *i.e.*, $x_i < x_j$ if and only if $i < j$.

We are going to divide the sequence $x_1, \ldots, x_m$ into subsets of size about $\epsilon n/(\chi+1)^2$. Let $U_k = \{i : x_i \in V_k\}$, $k = 1, \ldots, \chi$. Note that $|U_k| = n$. Let us write $U_k = \{i_1, \ldots, i_n\}$ in such a way that $i_1 < \cdots < i_n$. We define subsets A_j^k of U_k by

$$A_j^k = \left\{ i_{(j-1)\lceil \frac{\epsilon n}{(\chi+1)^2} \rceil + 1}, \ldots, i_{j\lceil \frac{\epsilon n}{(\chi+1)^2} \rceil} \right\},$$

where $j = 1, \ldots, t$ and $t = \lfloor n / \lceil \frac{\epsilon n}{(\chi+1)^2} \rceil \rfloor$. Thus we obtain $\tau = \chi t$ sets $A_j^k, k = 1, \ldots, \chi$, $j = 1, \ldots, t$, each of size $\lceil \epsilon n/(\chi+1)^2 \rceil$.

We define a partial ordering of subsets of integers as follows.

Definition 6. For two sets A and B of integers, we define $A \lhd B$ if and only if $\max A < \min B$.

To bound the number of dependent arcs in $\vec{G}$ we will need to count the χ-element sets of sets A_j^k with the property that all $\binom{\chi}{2}$ pairs of such sets are comparable under the relation $\lhd$. We start with the following claim.

Lemma 8.1. *Let α, β be positive integers. Let $\mathscr{A} = \{A_1, \ldots, A_\alpha\}$ and $\mathscr{B} = \{B_1, \ldots, B_\beta\}$ be two set systems of subsets of integers satisfying $A_1 \lhd \cdots \lhd A_\alpha$ and $B_1 \lhd \cdots \lhd B_\beta$. Consider a bipartite graph $\Gamma = (V_\Gamma, E_\Gamma)$ with $V_\Gamma = \mathscr{A} \cup \mathscr{B}$ and with $A \in \mathscr{A}$ and $B \in \mathscr{B}$ joined by an edge if and only if the sets A and B are not comparable under the relation $\lhd$. Then, $|E_\Gamma| \leqslant |V_\Gamma| - 1$.*

Proof. First observe that it is impossible to have four sets $A_{i_1}, A_{i_2} \in \mathscr{A}$, $B_{j_1}, B_{j_2} \in \mathscr{B}$ satisfying $i_1 < i_2, j_1 < j_2$, and both $\{A_{i_1}, B_{j_2}\}, \{A_{i_2}, B_{j_1}\} \in E_\Gamma$.

We proceed by induction on $v = |V_\Gamma|$. Clearly, in the base case $v = 2$ the number of edges is no more than $v - 1$.

Suppose the claim holds for any positive integer up to $v - 1$. Owing to the observation above at least one of the sets $\{\{A_\alpha, B_j\} \in E_\Gamma : j = 1, \ldots, \beta - 1\}$ and $\{\{A_i, B_\beta\} \in E_\Gamma : i = 1, \ldots, \alpha - 1\}$ must be empty. Without loss of generality, we may assume that the second set is empty. Set $\gamma = \max\{j \in \{1, \ldots, \beta - 1\} : B_j$ has an edge into the set $\{A_1, \ldots, A_{\alpha-1}\}\}$. Using the induction hypothesis, we get for the number of edges e of Γ

$$e \leqslant (\alpha - 1 + \gamma - 1) + (\beta - \gamma + 1) = \alpha + \beta - 1 = v - 1. \tag{8.1}$$

Thus, the claim follows. □

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Let Ψ be the following set: $\Psi = \{\{B_1,\dots,B_\chi\} : (\forall i \in \{1,\dots,\chi\})\ (\exists j \in \{1,\dots,t\})\ (\exists k \in \{1,\dots,\chi\})$ such that $B_i = A_j^k$ and each set B_i, $i = 1,\dots,\chi$, belongs to a different colour class of $G\}$.

Lemma 8.2. *Among all t^χ elements of Ψ there are at least $t^\chi - 2\binom{\chi}{2}t^{\chi-1}$ elements $\{B_1,\dots,B_\chi\}$ with all pairs $\{B_i,B_j\}$, $i,j \in [\chi]$, $i \neq j$, comparable by the relation $\lhd$.*

Proof. For each $k = 1,\dots,\chi$, let $W_k = \{A_j^k : j = 1,\dots,t\}$. Let us define a χ-partite auxiliary graph $\Gamma = (V_\Gamma, E_\Gamma)$ with $V_\Gamma = W_1 \cup \dots \cup W_\chi$ and with A,B forming an edge if and only if the sets A and B are not comparable by the relation $\lhd$.

By Claim 8.1 above, the number of edges $e(\Gamma)$ in Γ is at most $2\binom{\chi}{2}t$. Consider the following set of subgraphs on χ vertices of the graph Γ: $\Upsilon = \{H = (V(H),E(H)) : H$ is a subgraph of Γ, $|E(H)| \geqslant 1$, and for every $k = 1,\dots,\chi$, $|V(H) \cap W_k| = 1\}$. Every edge of Γ is contained in at most $t^{\chi-2}$ subgraphs $H \in \Upsilon$. Thus, the number of elements of Ψ containing at least one incomparable pair of sets is at most $e(\Gamma)t^{\chi-2} \leqslant 2\binom{\chi}{2}t^{\chi-1}$. Therefore, the claim follows. □

We call a χ-element set $\{B_1,\dots,B_\chi\} \in \Psi$ *good* if all pairs $\{B_i,B_j\}$, $i,j \in [\chi]$, $i \neq j$, are comparable under the relation $\lhd$. Let $\overline{\Psi}$ be the set of all good elements of Ψ. Note that $|\overline{\Psi}| \geqslant t^\chi - 2\binom{\chi}{2}t^{\chi-1}$ by Lemma 8.2.

Let $\{B_1,\dots,B_\chi\}$ be an element of the set $\overline{\Psi}$ such that $B_1 \lhd \dots \lhd B_\chi$. We call a pair $\{B_i,B_j\}, i<j$, *long* if and only if $j - i \geqslant g - 1$. Let *long*$(\overline{\Psi})$ denote the number of pairs $\{A,B\}$ of sets A_j^k such that there is an element $\{B_1,\dots,B_\chi\}$ of $\overline{\Psi}$ with $\{A,B\}$ being a long pair in $\{B_1,\dots,B_\chi\}$. By straightforward counting we get

$$\begin{aligned}\text{long}(\overline{\Psi}) &\geqslant \frac{1}{t^{\chi-2}} \sum_{\{B_1,\dots,B_\chi\}\in\overline{\Psi}} \text{the number of long pairs in } \{B_1,\dots,B_\chi\}\\ &\geqslant \frac{1}{t^{\chi-2}}(t^\chi - 2\binom{\chi}{2}t^{\chi-1})\frac{(\chi-g+1)(\chi-g+2)}{2}\\ &= \left(t^2 - 2\binom{\chi}{2}t\right)\binom{\chi-g+2}{2}.\end{aligned}$$

Using the lower bound on long$(\overline{\Psi})$ and the properties of the graph G (*cf.* Proposition 5.1), we can bound the number of dependent arcs $d(\vec{G})$ in $\vec{G}$:

$$d(\vec{G}) \geqslant \text{long}(\overline{\Psi}) \cdot \left(\left(1 - \frac{\epsilon}{(\chi+1)^2}\right)\frac{\epsilon^2 n^2}{(\chi+1)^4} n^{-1+\frac{1}{g-1}+\lambda} - \psi\right). \tag{8.2}$$

Indeed, let $\{A,B\}$ be a long pair. Then there is $\{B_1,\dots,B_\chi\} \in \overline{\Psi}$, $B_1 \lhd \dots \lhd B_\chi$, with $B_i = A$ and $B_j = B$, $j - i \geqslant g - 1$. The number of edges between sets B_i and B_j of size at least $\lceil \frac{\epsilon n}{(\chi+1)^2} \rceil$ from different colour classes of G is at least $\left(1 - \frac{\epsilon}{(\chi+1)^2}\right)\frac{\epsilon^2 n^2}{(\chi+1)^4} n^{-1+\frac{1}{g-1}+\lambda}$ and all of them but at most ψ are $(B_k)_{k=i+1}^{j-1}$-connected and thus dependent.

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The inequalities below hold under the assumption n being sufficiently large. Using (8.2) we get

$$d(\vec{G}) \geqslant \binom{\chi - g + 2}{2}\left[t^2 - 2\binom{\chi}{2}t\right]\left[\left(1 - \frac{\epsilon}{(\chi+1)^2}\right)\frac{\epsilon^2 n^2}{(\chi+1)^4}n^{-1+\frac{1}{g-1}+\lambda} - \psi\right].$$

Using the inequalities $\frac{(\chi+1)^2}{\epsilon} \geqslant t = \lfloor n/\lceil \frac{\epsilon n}{(\chi+1)^2}\rceil\rfloor \geqslant \frac{(\chi+1)^2}{\epsilon} - 2$, we get

$$\begin{aligned} d(\vec{G}) \geqslant \binom{\chi - g + 2}{2}&\left[\left(\frac{(\chi+1)^2}{\epsilon} - 2\right)^2 - \frac{2(\chi+1)^2}{\epsilon}\binom{\chi}{2}\right]\cdot n^{1+\frac{1}{g-1}+\lambda} \\ &\times\left[\left(1 - \frac{\epsilon}{(\chi+1)^2}\right)\frac{\epsilon^2}{(\chi+1)^4} - \frac{\psi}{n^{1+\frac{1}{g-1}+\lambda}}\right]. \end{aligned}$$

Using the facts $\psi = \psi(n) = \frac{1}{\ln n}\frac{\epsilon^2}{(\chi+1)^4}n^{1+\frac{1}{g-1}+\lambda}$ and $e(G) \leqslant \left(1 + \frac{\epsilon}{(\chi+1)^2}\right)\binom{\chi}{2}n^{1+\frac{1}{g-1}+\lambda}$ (*cf.* Proposition 5.1), we further get

$$\begin{aligned} d(\vec{G}) &\geqslant \binom{\chi - g + 2}{2}\frac{e(G)}{\binom{\chi}{2}}\left[\left(\frac{(\chi+1)^2}{\epsilon} - 2\right)^2 - 2\frac{(\chi+1)^2}{\epsilon}\binom{\chi}{2}\right]\frac{\epsilon^2}{(\chi+1)^4} \\ &\quad\times\frac{1}{1+\frac{\epsilon}{(\chi+1)^2}}\cdot\left[1 - \frac{\epsilon}{(\chi+1)^2} - \frac{1}{\ln n}\right] \\ &\geqslant \frac{\binom{\chi-g+2}{2}}{\binom{\chi}{2}}e(G)\left[1 - 4\frac{\epsilon}{(\chi+1)^2} + 4\frac{\epsilon^2}{(\chi+1)^4} - 2\binom{\chi}{2}\frac{\epsilon}{(\chi+1)^2}\right] \\ &\quad\times\left[1 - \frac{\epsilon}{(\chi+1)^2}\right]\left[1 - \frac{2\epsilon}{(\chi+1)^2}\right] \\ &\geqslant \frac{\binom{\chi-g+2}{2}}{\binom{\chi}{2}}e(G)\left[1 - (\chi^2+4)\frac{\epsilon}{(\chi+1)^2}\right]\left[1 - \frac{3\epsilon}{(\chi+1)^2}\right] \\ &\geqslant \frac{\binom{\chi-g+2}{2}}{\binom{\chi}{2}}e(G)[1-\epsilon]. \end{aligned}$$

This proves Proposition 4.1. □

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