Homogeneous variables in classical dynamics. By Professor P. A. M. Dirac, St John's College.

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1. *Introduction.*

The well-known methods of classical mechanics, based on the use of a Lagrangian or Hamiltonian function, are adequate for the treatment of nearly all dynamical systems met with in practice. There are, however, a few exceptional cases to which the ordinary methods are not immediately applicable. For example, the ordinary Hamiltonian method cannot be used when the momenta p_r, defined in terms of the Lagrangian function L by the usual formulae $p_r = \partial L/\partial \dot{q}_r$, are not independent functions of the velocities. A practical case of this kind is provided by the electromagnetic field, considered as a dynamical system with an infinite number of degrees of freedom, since the momentum conjugate to the scalar potential at any point vanishes identically*. Again, for the very simple example of the relativistic motion of a particle of zero rest-mass in field-free space, the Lagrangian function vanishes and the usual Lagrangian method is not applicable.

One can avoid having irregularities in the theory due to such exceptional cases by a reformulation of the scheme of equations with the use of homogeneous variables. No essential change of principle is involved in this reformulation, but a greater elegance is obtained, somewhat similar to that accompanying the introduction of homogeneous coordinates into geometry.

The homogeneity condition may be applied either to the velocities, as discussed in § 2, or to the momenta, as discussed in § 3. The main change when the velocities are made homogeneous is that the Hamiltonian function is replaced by a Hamiltonian equation, and the main change when the momenta are made homogeneous is that the Lagrangian function is replaced by a Lagrangian equation. In either case an extra coordinate must be introduced, which may be treated on the same footing as the previously existing ones. With combined homogeneous velocities and momenta there must be two such extra coordinates.

2. *Homogeneous velocities.*

We consider a dynamical system of n degrees of freedom, with coordinates q_r and velocities $\dot{q}_r$ $(r = 1, \ldots, n)$. There will be a Lagrangian function, L_1 say, which will be some function of the

* See Heisenberg and Pauli, *Zeits. f. Physik*, 56, p. 25 (1929).

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coordinates, velocities and perhaps the time t, and the Lagrangian equations of motion will be

$$\frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_r} = \frac{\partial L_1}{\partial q_r}. \tag{1}$$

We now count the time t as an extra coordinate q_0. The corresponding velocity $\dot{q}_0$ will be identically equal to unity. We can introduce $\dot{q}_0$ as a factor into certain places in L_1 in such a way as to make it homogeneous of the first degree in all the velocities $\dot{q}_0, \dot{q}_1, \ldots, \dot{q}_n$. In symbols, if the new Lagrangian function obtained in this way is $L(q_0, q_r, \dot{q}_0, \dot{q}_r)$ and the old one is $L_1(t, q_r, \dot{q}_r)$, we have

$$L(q_0, q_r, \dot{q}_0, \dot{q}_r) = \dot{q}_0 L_1\left(q_0, q_r, \frac{\dot{q}_r}{\dot{q}_0}\right).$$

Let us see what the equations of motion are in terms of the new Lagrangian function. We have immediately

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} = \frac{\partial L}{\partial q_r}, \tag{2}$$

since these equations reduce to (1) when one puts $\dot{q}_0 = 1$. On account of L being homogeneous of the first degree in the velocities, we obtain from Euler's theorem *

$$\dot{q}_0 \frac{\partial L}{\partial \dot{q}_0} + \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} = L. \tag{3}$$

Putting $\dot{q}_0 = 1$ and taking the time derivative of this equation, we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_0} + \ddot{q}_r \frac{\partial L}{\partial \dot{q}_r} + \dot{q}_r \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} = \frac{\partial L}{\partial q_0} + \frac{\partial L}{\partial q_r}\dot{q}_r + \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r,$$

which reduces, with the help of (2), to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_0} = \frac{\partial L}{\partial q_0}.$$

This equation may be combined with (2) to give the set

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\mu} = \frac{\partial L}{\partial q_\mu}, \qquad \mu = 0, 1, \ldots, n. \tag{4}$$

We now have an equation of motion for q_0 of the same form as those for $q_1, \ldots, q_n$, but we have not yet got complete symmetry between q_0 and the other q's on account of the fact that our operation of total differentiation refers to a variable t which is just equal

* We use the summation convention that a term with a repeated suffix is to be summed for all allowed values of that suffix. The allowed values for r are 1, ..., n; those for μ are 0, 1, ..., n; those for a are 1, ..., n, $n+1$; and those for k are 0, 1, ..., n, $n+1$.

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to q_0. We may, however, introduce a new independent variable t^*, which can be any function of the coordinates $q_0, q_1, \ldots, q_n$, and use it instead of t. The form of the equations of motion (4) will not be altered thereby. One can verify this by direct transformation, or, more elegantly, by showing that the equations of motion (4) may be expressed in the form of an action principle which involves no reference to any independent variable and treats q_0 on the same footing as the other coordinates.

We introduce the action integral

$$A = \int_{t_1}^{t_2} L dt. \tag{5}$$

The coordinates $q_0, q_1, \ldots, q_n$ are here considered as functions of the independent variable t and we make an arbitrary small variation δq_μ in each of them. Thus the equation $\dot{q}_0 = 1$ does not hold for the varied motion. The values of all the coordinates $q_0, q_1, \ldots, q_n$ are kept fixed at the two ends of the domain of integration in (5), but the end-values t_1 and t_2 of the independent variable are allowed to vary. If δt_1, δt_2 are the amounts by which they vary, we have

$$\begin{aligned}
\delta A &= \delta \int_{t_1}^{t_2} L dt \\
&= \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_\mu} \delta q_\mu + \frac{\partial L}{\partial \dot{q}_\mu} \delta \dot{q}_\mu \right\} dt + (L)_{t_2} \delta t_2 - (L)_{t_1} \delta t_1 \\
&= \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_\mu} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\mu} \right\} \delta q_\mu dt + \left(\frac{\partial L}{\partial \dot{q}_\mu} \delta q_\mu \right)_{t_2} - \left(\frac{\partial L}{\partial \dot{q}_\mu} \delta q_\mu \right)_{t_1} \\
&\qquad + (L)_{t_2} \delta t_2 - (L)_{t_1} \delta t_1. \qquad (6)
\end{aligned}$$

Now $(\delta q_\mu)_{t_2}$ means the variation in q_μ for a fixed value of t near the point t_2 and thus

$$(\delta q_\mu)_{t_2} = -(\dot{q}_\mu)_{t_2} \delta t_2.$$

Similarly
$$(\delta q_\mu)_{t_1} = -(\dot{q}_\mu)_{t_1} \delta t_1.$$

Hence the terms outside the integral on the right-hand side of (6) are equal to

$$\left(L - \frac{\partial L}{\partial \dot{q}_\mu} \dot{q}_\mu \right)_{t_2} \delta t_2 - \left(L - \frac{\partial L}{\partial \dot{q}_\mu} \dot{q}_\mu \right)_{t_1} \delta t_1,$$

which vanishes on account of the homogeneity condition (3). We are left with

$$\delta A = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q_\mu} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\mu} \right\} \delta q_\mu \, dt,$$

which shows that the equations of motion (4) are equivalent to the condition that A is stationary for small variations of the q_μ satisfying the end conditions stated.

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On account of (3), the action integral (5) may be written

$$A = \int \frac{\partial L}{\partial \dot{q}_\mu} \dot{q}_\mu \, dt = \int \frac{\partial L}{\partial \dot{q}_\mu} \, dq_\mu .$$

This expression is invariant under any transformation of the independent variable, from t to t^*, since the $\partial L/\partial \dot{q}_\mu$ are homogeneous functions of the velocities of degree zero. Also, the end conditions in the variation problem do not involve the independent variable. Thus the whole action principle may be considered as not involving any independent variable, but as involving only a set of mutually dependent coordinates q_μ.

It follows that the equations of motion (4), which are equivalent to the action principle, must be invariant under a transformation from t to t^*. These equations may, in fact, be written in a form in which they do not involve any independent variable by a suitable use of differentials. We put

$$dL = \frac{\partial L}{\partial \dot{q}_\mu} \, dq_\mu ,$$

and consider dL as a function of the $2n+2$ quantities q_μ, dq_μ. It will be the same function of these quantities as L is of q_μ, $\dot{q}_\mu$. We may then write (4) in the form

$$d \frac{\partial dL}{\partial dq_\mu} = \frac{\partial dL}{\partial q_\mu} . \tag{7}$$

We now introduce the momenta, defined by

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} . \tag{8}$$

In the case when the independent variable is $t = q_0$, the momenta $p_1, \ldots, p_n$ are just the same as the momenta in the ordinary theory with non-homogeneous velocities, while

$$p_0 = \frac{\partial L}{\partial \dot{q}_0} = L - \dot{q}_r p_r ,$$

from (3). Thus $-p_0$ is just the Hamiltonian function of the ordinary theory, as is to be expected on account of its being the conjugate to the time. The further work in this section will apply to the case of a general independent variable t^*.

Each momentum p_μ is a homogeneous function of the velocities of degree zero and is thus a function only of the ratios of the velocities. The $n+1$ momenta cannot all be independent functions of the n ratios of the velocities, so there must be some relation between them of the type

$$\mathfrak{H}(p, q) = 0. \tag{9}$$

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This relation is the Hamiltonian equation. In special cases there may be two or more independent relations of this kind, say

$$\mathfrak{H}_a(p, q) = 0, \quad \mathfrak{H}_b(p, q) = 0, \ldots. \tag{10}$$

There would then be two or more Hamiltonian equations, to be treated on the same footing.

Equation (3) may be written in the form

$$\dot{q}_\mu p_\mu - L = 0. \tag{11}$$

If we make arbitrary independent variations δq_μ, $\delta \dot{q}_\mu$, giving rise to variations δp_μ according to (8), we get

$$\dot{q}_\mu \delta p_\mu + p_\mu \delta \dot{q}_\mu - \frac{\partial L}{\partial q_\mu} \delta q_\mu - \frac{\partial L}{\partial \dot{q}_\mu} \delta \dot{q}_\mu = 0,$$

which reduces to

$$\dot{q}_\mu \delta p_\mu - \frac{\partial L}{\partial q_\mu} \delta q_\mu = 0.$$

This equation shows that the p_μ and q_μ cannot all be varied independently. If (9) is the only relation between them, we have

$$\dot{q}_\mu = \rho \frac{\partial \mathfrak{H}}{\partial p_\mu}, \quad -\frac{\partial L}{\partial q_\mu} = \rho \frac{\partial \mathfrak{H}}{\partial q_\mu},$$

where ρ is an undetermined multiplier. From (4) these equations give

$$\dot{q}_\mu = \rho \frac{\partial \mathfrak{H}}{\partial p_\mu}, \quad \dot{p}_\mu = -\rho \frac{\partial \mathfrak{H}}{\partial q_\mu}, \tag{12}$$

which are the Hamiltonian equations of motion. When there are several Hamiltonian equations (10), there must be a corresponding number of undetermined multipliers ρ_a, ρ_b, ... and the equations of motion will be of the form

$$\left.\begin{aligned} \dot{q}_\mu &= \rho_a \frac{\partial \mathfrak{H}_a}{\partial p_\mu} + \rho_b \frac{\partial \mathfrak{H}_b}{\partial p_\mu} + \ldots = \frac{\partial}{\partial p_\mu}(\rho_a \mathfrak{H}_a + \rho_b \mathfrak{H}_b + \ldots) \\ \dot{p}_\mu &= -\rho_a \frac{\partial \mathfrak{H}_a}{\partial q_\mu} - \rho_b \frac{\partial \mathfrak{H}_b}{\partial q_\mu} - \ldots = -\frac{\partial}{\partial q_\mu}(\rho_a \mathfrak{H}_a + \rho_b \mathfrak{H}_b + \ldots) \end{aligned}\right\}. \tag{13}$$

3. *Homogeneous momenta.*

We go back to the non-homogeneous Lagrangian function L_1 that we had at the beginning of the preceding section and proceed to handle it in a different way. We introduce a new velocity $\dot{q}_{n+1}$, defined to be equal to the Lagrangian function L_1, so that we have the equation

$$L_1(t, q_r, \dot{q}_r) - \dot{q}_{n+1} = 0. \tag{14}$$

This equation, written in the more general form

$$\Omega(t, q_r, q_{n+1}, \dot{q}_r, \dot{q}_{n+1}) = 0, \tag{15}$$

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in which its left-hand side has been multiplied by an arbitrary factor $\chi(t, q_r, q_{n+1}, \dot{q}_r, \dot{q}_{n+1})$, we call the Lagrangian equation and proceed to use instead of the Lagrangian function as the basis of our dynamical theory.

From (1) we obtain

$$\frac{d}{dt}\frac{\partial(\lambda\mathfrak{L})}{\partial\dot{q}_r} = \frac{\partial(\lambda\mathfrak{L})}{\partial q_r},$$

where λ is the reciprocal of the factor χ that was used in passing from (14) to (15). With the same value of λ we have also

$$\frac{d}{dt}\frac{\partial(\lambda\mathfrak{L})}{\partial\dot{q}_{n+1}} = \frac{\partial(\lambda\mathfrak{L})}{\partial q_{n+1}},$$

both sides being zero. These equations may be combined to give

$$\frac{d}{dt}\frac{\partial(\lambda\mathfrak{L})}{\partial\dot{q}_a} = \frac{\partial(\lambda\mathfrak{L})}{\partial q_a}, \qquad a = 1, \ldots, n, n+1,$$

which may also be written, on account of (15),

$$\frac{d}{dt}\left(\lambda\frac{\partial\mathfrak{L}}{\partial\dot{q}_a}\right) = \lambda\frac{\partial\mathfrak{L}}{\partial q_a}. \tag{16}$$

We may suppose we have a dynamical system specified merely by a Lagrangian equation (15), the left-hand side of this equation being undetermined to the extent of an arbitrary factor. Then λ is an undetermined multiplier, of the same nature as the ρ's in (12) and (13), and (15) and (16) provide us with $n+2$ equations to determine the variation with time of the $n+2$ unknowns $q_1, \ldots, q_n, q_{n+1}, \lambda$. These are the Lagrangian equations of motion of the present method.

The new coordinate q_{n+1} is on the same footing as the original ones $q_1, \ldots, q_n$, except for the fact that the Lagrangian equation does not (effectively) involve the coordinate q_{n+1}, but only the corresponding velocity $\dot{q}_{n+1}$. If, however, there are any cyclic coordinates among the set $q_1, \ldots, q_n$, these will also not be involved in the Lagrangian equation and will be on just the same footing as q_{n+1}. The passage from a Lagrangian function to a Lagrangian equation thus involves the introduction of a new cyclic coordinate. It would be permissible to consider a dynamical system specified by a Lagrangian equation with no cyclic variable, so that there would be no way of passing to a Lagrangian function by a reversal of the procedure at the beginning of this section.

For our dynamical system specified merely by a Lagrangian equation (15), we define the momenta by

$$p_a = \lambda\frac{\partial\mathfrak{L}}{\partial\dot{q}_a}. \tag{17}$$

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On account of the undetermined nature of λ, only the ratios of the momenta are thus defined and all our equations have to be homogeneous in the momenta. We also define the Hamiltonian function by

$$H = \dot{q}_a p_a. \tag{18}$$

This equation should be compared with (11) in the homogeneous velocity theory.

We shall assume that the $n+1$ momenta p_a are independent functions of the $n+2$ variables $\dot{q}_a$, λ, of which $n+2$ variables only $n+1$ are independent on account of the relation (15) between the $\dot{q}_a$. (The case when this is not so is best treated with combined homogeneous velocities and momenta, as in the next section.) We can then solve for the $\dot{q}_a$ and λ and express each of them in terms of the momenta p_a (together with the coordinates q_a, of course). λ must be homogeneous of the first degree in the momenta and each $\dot{q}_a$ must be homogeneous of degree zero in the momenta. We can substitute these expressions for the $\dot{q}_a$ in (18) and obtain the Hamiltonian function as a function of the coordinates and momenta, which is homogeneous of the first degree in the momenta.

The above assumption means that we can make independent variations δp_a, δq_a in all the momenta and coordinates, consistent with variations $\delta\dot{q}_a$, $\delta\lambda$ in the velocities and λ which do not disturb the equation $\mathfrak{L} = 0$, i.e. such that

$$0 = \delta\mathfrak{L} = \frac{\partial\mathfrak{L}}{\partial q_a}\delta q_a + \frac{\partial\mathfrak{L}}{\partial\dot{q}_a}\delta\dot{q}_a. \tag{19}$$

Now from (18)

$$\delta H = \dot{q}_a\delta p_a + p_a\delta\dot{q}_a. \tag{20}$$

Multiplying (19) by λ and subtracting from (20), we obtain, with the help of (17),

$$\delta H = \dot{q}_a\delta p_a - \lambda\frac{\partial\mathfrak{L}}{\partial q_a}\delta q_a,$$

which gives, with the help of (16),

$$\delta H = \dot{q}_a\delta p_a - \dot{p}_a\delta q_a.$$

Hence

$$\dot{q}_a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q_a}. \tag{21}$$

These Hamiltonian equations of motion are of precisely the usual type. The only way in which the fact that we are working from a Lagrangian equation instead of a Lagrangian function shows itself is through the Hamiltonian function being homogeneous of the first degree in the momenta.

The theory of homogeneous momenta is not on quite the same footing physically as that of homogeneous velocities. The treatment

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of the time as an extra coordinate q_0 is a very natural thing physically, and is indeed demanded by relativity. On the other hand q_{n+1} appears to have no physical significance and its introduction and treatment as an extra coordinate are of no value physically. Again, while every dynamical system may be treated both with and without homogeneous velocities, there are certain special dynamical systems which can be treated only with homogeneous momenta. If dynamical systems are considered as the continual unfolding of contact transformations, for one of these special systems the contact transformation will be of the homogeneous type, instead of the non-homogeneous type usual in dynamics. This aspect of homogeneous momentum theory has been discussed by Eisenhart*.

4. *Combined homogeneous velocities and momenta.*

We can combine the methods of the two preceding sections to obtain the most general and symmetrical form of the equations of dynamics. We then start with a Lagrangian equation involving a number of coordinates q_k and velocities $\dot{q}_k$ $(k = 0, 1, \ldots, n+1)$, which has to be homogeneous in the velocities, but need not be of the first degree. (The degree, of course, is irrelevant in the case of an equation.) To be perfectly general we should allow the possibility of there being two or more independent Lagrangian equations, in the same way in which there could be two or more independent Hamiltonian equations in § 2. Each Lagrangian equation must be homogeneous in the velocities.

Let us suppose that we have m independent Lagrangian equations

$$\mathfrak{L}_x = 0, \quad \mathfrak{L}_y = 0, \ldots. \tag{22}$$

The Lagrangian equations of motion are

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}(\lambda_x \mathfrak{L}_x + \lambda_y \mathfrak{L}_y + \ldots) = \frac{\partial}{\partial q_k}(\lambda_x \mathfrak{L}_x + \lambda_y \mathfrak{L}_y + \ldots),$$

i.e.

$$\frac{d}{dt}\left(\lambda_x \frac{\partial \mathfrak{L}_x}{\partial \dot{q}_k} + \lambda_y \frac{\partial \mathfrak{L}_y}{\partial \dot{q}_k} + \ldots\right) = \lambda_x \frac{\partial \mathfrak{L}_x}{\partial q_k} + \lambda_y \frac{\partial \mathfrak{L}_y}{\partial q_k} + \ldots. \tag{23}$$

In (22) and (23) we have $m+n+2$ equations to determine the variation with time of the $m+n+2$ quantities $q_0, q_1, \ldots, q_{n+1}$, $\lambda_x, \lambda_y, \ldots$.

The momenta are defined by

$$\begin{aligned} p_k &= \frac{\partial}{\partial \dot{q}_k}(\lambda_x \mathfrak{L}_x + \lambda_y \mathfrak{L}_y + \ldots) \\ &= \lambda_x \frac{\partial \mathfrak{L}_x}{\partial \dot{q}_k} + \lambda_y \frac{\partial \mathfrak{L}_y}{\partial \dot{q}_k} + \ldots. \end{aligned} \tag{24}$$

* Eisenhart, *Ann. of Math.* 30, p. 211 (1929).

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The $n+2$ quantities p_k may be considered as functions of the $n+m+2$ quantities $\dot{q}_k$, λ_a, $\lambda_b, \ldots$, of which only $n+2$ are independent on account of (22). The p_k cannot all be independent functions of these variables. One can see this most easily by observing that, if the $\mathfrak{L}$'s are each of the first degree in the velocities, then the p_k will be homogeneous of degree zero in the velocities and thus functions only of the ratios of the velocities, and of these ratios, together with the quantities λ, only $n+1$ are independent. If the $\mathfrak{L}$'s are not of the first degree it cannot make any difference, owing to each $\mathfrak{L}$ in (24) being multiplied by an arbitrary factor λ.

There must therefore be a connection between the p_k of the type

$$\mathfrak{H}(p, q) = 0. \tag{25}$$

There may possibly be two or more independent connections of this type, say

$$\mathfrak{H}_a(p, q) = 0, \quad \mathfrak{H}_b(p, q) = 0, \ldots. \tag{26}$$

These are the Hamiltonian equations. Equation (25) or equations (26) may be regarded as the result of the elimination of the $\dot{q}$'s and λ's from (24) and (22) and it follows from the form of (24) that they must each be homogeneous in the momenta. Apart from this they are similar to (9) and (10).

From (24) we have

$$\begin{aligned} \dot{q}_k p_k &= \lambda_x \dot{q}_k \frac{\partial \mathfrak{L}_x}{\partial \dot{q}_k} + \lambda_y \dot{q}_k \frac{\partial \mathfrak{L}_y}{\partial \dot{q}_k} + \ldots \\ &= \lambda_x l_x \mathfrak{L}_x + \lambda_y l_y \mathfrak{L}_y + \ldots, \end{aligned}$$

where $l_x, l_y, \ldots$ are the degrees of $\mathfrak{L}_x, \mathfrak{L}_y, \ldots$ in the velocities. Hence from (22)

$$\dot{q}_k p_k = 0, \tag{27}$$

an equation which may be compared with (11) and (18).

Suppose that we make variations $\delta q_k, \delta \dot{q}_k$ which are arbitrary except for the conditions that equations (22) must hold also for the varied values, so that

$$\frac{\partial \mathfrak{L}_x}{\partial q_k} \delta q_k + \frac{\partial \mathfrak{L}_x}{\partial \dot{q}_k} \delta \dot{q}_k = 0, \quad \frac{\partial \mathfrak{L}_y}{\partial q_k} \delta q_k + \frac{\partial \mathfrak{L}_y}{\partial \dot{q}_k} \delta \dot{q}_k = 0, \ldots.$$

Multiplying these equations by $\lambda_x, \lambda_y, \ldots$ respectively and adding, we get

$$\left(\lambda_x \frac{\partial \mathfrak{L}_x}{\partial q_k} + \lambda_y \frac{\partial \mathfrak{L}_y}{\partial q_k} + \ldots\right) \delta q_k + \left(\lambda_x \frac{\partial \mathfrak{L}_x}{\partial \dot{q}_k} + \lambda_y \frac{\partial \mathfrak{L}_y}{\partial \dot{q}_k} + \ldots\right) \delta \dot{q}_k = 0.$$

With the help of (23) and (24) this gives

$$\dot{p}_k \delta q_k + p_k \delta \dot{q}_k = 0.$$

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But from (27)

$$\dot{q}_k\,\delta p_k + p_k\,\delta\dot{q}_k = 0.$$

Hence

$$\dot{q}_k\,\delta p_k - \dot{p}_k\,\delta q_k = 0.$$

This shows that the p_k and q_k cannot be independent and, if (26) gives the relations between them,

$$\left.\begin{aligned} \dot{q}_k &= \rho_a \frac{\partial \mathfrak{H}_a}{\partial p_k} + \rho_b \frac{\partial \mathfrak{H}_b}{\partial p_k} + \ldots, \\ \dot{p}_k &= -\rho_a \frac{\partial \mathfrak{H}_a}{\partial q_k} - \rho_b \frac{\partial \mathfrak{H}_b}{\partial q_k} - \ldots. \end{aligned}\right\} \tag{28}$$

These are the Hamiltonian equations of motion.

It should be noticed that there is a certain amount of symmetry between the velocities and the momenta. Given any dynamical system, we could obtain a dual system by interchanging the velocities and momenta, at the same time interchanging the Hamiltonian and Lagrangian equations. This notion of duality is probably not very significant however, since it is not invariant under a transformation of the coordinates, owing to the fact that the momenta transform covariantly and the velocities contravariantly.

5. *Examples.*

As a simple example of the foregoing methods we may take the relativistic motion of a particle of charge e and rest-mass m in an electromagnetic field described by the potentials A_0, A_1, A_2, A_3. There are three dynamical coordinates, the three Cartesian co-ordinates x_1, x_2, x_3, and three conjugate momenta p_1, p_2, p_3. The ordinary Hamiltonian function, equal to the total energy (kinetic plus potential), is

$$H = c\,[m^2 c^2 + \{p_r - (e/c)\,A_r\}^2]^{\frac{1}{2}} + eA_0, \qquad r = 1, 2, 3. \tag{29}$$

We have

$$\dot{x}_r = \frac{\partial H}{\partial p_r} = \frac{c\,\{p_r - (e/c)\,A_r\}}{[m^2 c^2 + \{p_r - (e/c)\,A_r\}^2]^{\frac{1}{2}}}. \tag{30}$$

Introducing the proper-time element ds defined by

$$c^2\,ds^2 = c^2\,dt^2 - dx_r^2,$$

we have

$$\left(\frac{ds}{dt}\right)^2 = 1 - \frac{\dot{x}_r^2}{c^2} = \frac{m^2 c^2}{m^2 c^2 + \{p_r - (e/c)\,A_r\}^2},$$

from (30), so that (30) may be written

$$m\frac{dx_r}{ds} = p_r - \frac{e}{c}A_r. \tag{31}$$

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The Lagrangian function L_1 has the value

$$\begin{aligned} L_1 &= p_r \dot{x}_r - H \\ &= \left(m\frac{dx_r}{ds} + \frac{e}{c}A_r\right)\frac{dx_r}{dt} - mc^2\frac{dt}{ds} - eA_0 \\ &= -mc^2\frac{ds}{dt} + \frac{e}{c}A_r\frac{dx_r}{dt} - eA_0 \\ &= -mc\,(c^2 - \dot{x}_r^2)^{\frac{1}{2}} + (e/c)\,A_r\dot{x}_r - eA_0. \end{aligned} \tag{32}$$

This treatment is quite unsymmetrical between the three space variables x_1, x_2, x_3 and the time variable t. We can get a symmetrical treatment suitable for relativity by the method of homogeneous velocities. We count the time t as an extra coordinate x_0 and introduce $\dot{x}_0$ into the Lagrangian function (32) to make it homogeneous of the first degree in the velocities. This gives us the new Lagrangian function

$$L = -mc\,(c^2\dot{x}_0^2 - \dot{x}_r^2)^{\frac{1}{2}} + (e/c)\,A_r\dot{x}_r - eA_0\dot{x}_0, \tag{33}$$

whose relativistic form may be seen if we write its differential

$$dL = -mc\,(c^2\,dx_0^2 - dx_r^2)^{\frac{1}{2}} + (e/c)\,A_r\,dx_r - eA_0\,dx_0.$$

We now have an extra momentum p_0, whose value is

$$p_0 = \frac{\partial L}{\partial \dot{x}_0} = -mc^2\frac{dx_0}{ds} - eA_0. \tag{34}$$

This equation is to be combined with (31). From (31) and (34) we may eliminate the velocities and get

$$(p_0 + eA_0)^2 - c^2\{p_r - (e/c)\,A_r\}^2 - m^2c^4 = 0.$$

This is the Hamiltonian equation of the system. It is the same equation as (29) with $-p_0$ for H.

Let us now take the case when there is no field, $A_\mu = 0$, and the particle has zero rest-mass, $m = 0$. The Lagrangian function (32) or (33) then vanishes and the usual Lagrangian method fails. We have to work from a Lagrangian equation instead. If we work also with homogeneous velocities, this equation will be

$$\Omega \equiv c^2\dot{x}_0^2 - \dot{x}_r^2 = 0. \tag{35}$$

The momenta will now be

$$p_0 = \lambda\frac{\partial\Omega}{\partial\dot{x}_0} = 2\lambda c^2\dot{x}_0,$$

$$p_r = \lambda\frac{\partial\Omega}{\partial\dot{x}_r} = -2\lambda\dot{x}_r,$$

so that, with the help of (35), we get as Hamiltonian equation

$$p_0^2 - c^2p_r^2 = 0, \tag{36}$$

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which is homogeneous in the momenta. The equations of motion are given either by (23) applied to the Lagrangian equation (35) or by (28) applied to the Hamiltonian equation (36).

The method of homogeneous momenta may be applied to the general case by the introduction of a fifth coordinate x_4. We then work from the Lagrangian equation

$$\mathfrak{L} \equiv -mc\,(c^2\dot{x}_0^2 - \dot{x}_r^2)^{\frac{1}{2}} + (e/c)\,A_r\,\dot{x}_r - eA_0\,\dot{x}_0 - \dot{x}_4 = 0.$$

The momenta are

$$p_0 = \lambda\frac{\partial\mathfrak{L}}{\partial\dot{x}_0} = -\lambda\left(mc^2\frac{dx_0}{ds} + eA_0\right),$$

$$p_r = \lambda\frac{\partial\mathfrak{L}}{\partial\dot{x}_r} = \lambda\left(m\frac{dx_r}{ds} + \frac{e}{c}A_r\right),$$

$$p_4 = \lambda\frac{\partial\mathfrak{L}}{\partial\dot{x}_4} = -\lambda,$$

from which we can obtain the homogeneous Hamiltonian equation

$$(p_0 - eA_0\,p_4)^2 - c^2\{p_r + (e/c)\,A_r\,p_4\}^2 - m^2c^4\,p_4^2 = 0.$$

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