Enumerative Formulae for Surfaces. By L. ROTH, B.A., Clare College.

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The formulae considered in this paper deal with the multiple tangent lines and planes to surfaces in three dimensions; the functional method employed is that of a previous paper*, in which a more restricted discussion was given for surfaces in higher space.

In the present work the surfaces are assumed to possess a double curve containing pinch-points and triple points; and it is only for lack of the necessary special cases that we have not considered surfaces which possess a cuspidal curve also and, in fact, all other singularities of a general type. Incidentally, the formulae obtained therefore apply to the general surface without point-singularities; we have thus a new and, for the most part, a simpler method of deriving these results, which differs from the algebraic method of Salmon, the descriptive method of Zeuthen, and the enumerative process developed by Schubert.

The subject-matter is divided into two parts. The first part, dealing with multiple tangent lines and allied problems, is, in its present general application, new; for the non-singular surface, many of the results are due to Salmon† and Schubert‡; for the special case of *scrolls*, the theory has been further developed by James§ in his first paper on the subject; to this work the present paper is greatly indebted for suggestions and for comparison of results. It will appear, however, that the problems here considered have an interest and difficulties of their own, besides including in particular most of the formulae given by James.

The second part is concerned with singular tangent planes to surfaces. A complete discussion has up to the present been given only for surfaces without singularities, and is to be found in various papers by Cayley, Salmon, Schubert and Basset‖. The consideration of surfaces possessing singularities was begun by Cayley and Salmon and greatly extended by Zeuthen¶; further results, of

* *Proc. Camb. Phil. Soc.* 25 (1929), 390—406; this is referred to as F.

† See Salmon-Rogers, *Analytic Geometry of Three Dimensions*, II (1915); this is referred to later as Salmon.

‡ Schubert, *Math Ann.* 10 (1876), 98; *ibid.* 11 (1877), 347; 12 (1877), 180. See also *Kalkül der Abzählenden Geometrie* (1879).

§ James, *Trans. Camb. Phil. Soc.* 23 (1925), 201; this is referred to as J.

‖ Cayley, *Coll. Papers*, VI, 359; Schubert, *op. cit.*; Basset, *Geometry of Surfaces*, Ch. IX (referred to as Basset).

¶ Cayley, *Coll. Papers*, VI, 582, XI, 225; Zeuthen, *Math. Ann.* 10 (1876), 446 (this paper is referred to as Zeuthen).

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uncertain validity, have been given by Basset* for surfaces with isolated nodes. Some of the present formulae are implied in the very general equations given by Zeuthen, which are obtained in a completely different manner; it may be added that the method of the present paper is very much simpler in application than that employed by Zeuthen for the same purpose, and yields its results more readily, the majority of them from first principles. A few formulae are due originally to Cayley, and these are acknowledged in the sequel. To Cayley† is due also the first hint at a functional method in this subject, in the assumption of a solution involving a series of linear terms in the characters of a surface, the coefficients of which are to be determined from known special cases.

I.

1. *Introduction.* In a previous paper already referred to, it was shown that the enumerative properties of a general surface Φ in [4] could be discussed by the Cayley-Severi functional method, provided that the following were taken as additive characters of Φ:

n, the order; a, the rank of a space section; j, the number of apparent pinch-points; and m, the class.

Now, if Φ is projected from an arbitrary vertex O on to ordinary space, the projection F has the same characters as Φ (j now being the number of pinch-points on the double curve). Planes and spaces through O which have various relations with Φ project into lines and planes having certain relations with F; the latter results could, of course, be obtained in their four-dimensional form, though they are of much greater interest in their present application. When in the functional process we associate two arbitrary surfaces Φ, Φ' in [4], the cone of chords from O consists of chords drawn to Φ and Φ' and common secants to the two surfaces. Hence, if the functional method is to be applied to two surfaces F, F' in [3], their complete intersection must be regarded as part of the composite double curve, in addition to the double curves on F and F'.

1·1. Since any surface in [3] may be regarded as the projection of a surface in [4] (which may of course be identical with itself), it follows that we can apply the functional method to any surface F whose point-singularities consist of a double curve containing pinch-points and triple points. The greater part of the present paper is an application of this method.

1·2. *Note.* We could, theoretically, extend the discussion to surfaces possessing all ordinary point-singularities, by taking into account an appropriate number of additive characters. For a

* Basset, *Quart. Journal*, 42 (1911), 225.

† Cayley, *Coll. Papers*, VI, 347.

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cuspidal curve, these would be the order and rank, and the number of points, of various types, in which it meets the double curve*. For accidental singularities, we could assume characters C and B, representing the number of conic nodes and binodes of F. In practice, however, it would be necessary to know the solution of the results about to be given for as many particular surfaces as there are characters in the functional equations; and this knowledge is at present unobtainable.

2. *Solution of the equations.* The notation and methods of dealing with the functional equations are the same as in the previous work, to which the reader is referred. For present purposes it is important to note the following features of the solutions. A general scroll depends on two characters (n, a) and it is found, as before, that the formulae obtained do not in general agree with the corresponding results for scrolls. In formulae which give the *order* of various scrolls, there is a consistent discrepancy in the coefficient of n alone; in formulae derived from these, the disagreement is in non-linear as well as linear terms. In the present work, all such scroll terms will be distinguished by a coefficient ϵ which is to be taken as zero when the formulae are applied to surfaces.

2·1. It is also found that formulae of certain types have characteristic solutions which may evidently be anticipated. Thus, for the order of the curve of contact of a scroll with F, the linear terms of the solution are always independent of j and m, and, in the solution of the dual problem, the term n does not appear. The formula for the class of a congruence is likewise independent of j and m, and dually, the formula for the order is independent of n. As instances of this phenomenon we may quote the results of §§ 5, 6·1, 8, 10, 12·1, 30·4, 34·2. Since no reason has been advanced for its occurrence, in no case has it been taken for granted unless there are other grounds for this assumption.

2·2. The accompanying table gives the surfaces which are required as special cases in the solution of the equations; several of these have been given before in a different notation†. The characters l and λ represent the number of ordinary and nodal lines which lie on the surface. The surfaces are all supposed to be the most general of their type: f_3 is the general cubic surface, F_4' the Steiner surface, J_6 the projection of the polo-surface in [4]; ϕ_n is the projection of the del Pezzo surface in $[n]$, and f_n is the general non-singular surface in ordinary space. Finally, F_4, F_5' and F_5'' are the most general surfaces of their order with the nodal lines as shown.

* See Zeuthen, p. 450; Salmon, p. 314.

† See F. §§ 1, 13; the surface F_7, which is described in § 1, has one improper node, not two, as stated.

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n	a	j	m	l	λ
f_3	6	0	12	27	0
F_4	10	4	20	16	1
F_4'	6	6	3	0	3
ϕ_4	8	4	12	16	0
ϕ_5	10	8	12	10	0
F_5	12	8	20	14	0
F_5'	18	6	57	0	1
F_5''	16	12	34	13	2
J_6	16	14	27	10	0
f_n	$n(n-1)$	0	$n(n-1)^2$	—	0
ϕ_n	$2n$	$4(n-3)$	12	—	0

3. *The principle of redundancy.* If a surface F contains l simple lines, these will occur redundantly in problems concerning the order of a curve or a finite number of lines. It may be shown, as in F. § 4, that each of the lines counts k times in the solution, where k, the "line constant," is independent of F and depends only upon the problem.

3·1. If F contains λ nodal lines *in general position*, these will also occur redundantly in the above problems and, by symmetry, each will count $d^{(n)}$ times, say, in the solution. This number $d^{(n)}$ depends only on the problem and the order of F; for the acquisition of an arbitrary nodal line causes a reduction of 2 in the rank and $7n - 12$

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in the class*, and an addition of $2n-4$ pinch-points. We shall call $d^{(n)}$ the "nodal line constant" for the problem in question.

3·2. The determination of k and $d^{(n)}$ in the sequel is generally performed as follows: k is found by substituting the characters of ϕ_4 in a formula which should give a zero result; the number obtained is then $16k$. The constant $d^{(4)}$ is found by applying the scroll formula to the scroll (4, 8) which contains two skew nodal lines; in a case where the result should be zero, we thus obtain $2d^{(4)}$. A verification of the formula is then provided by the surface F_4, for which $l=16$, $\lambda=1$. To obtain $d^{(5)}$ we apply the formula to F_5' for which $\lambda=1$, and then use F_5'' as a verification; in the case of a zero result the answer should be $13k+2d^{(5)}$.

3·3. Throughout most of the work there is an unfortunate lack of special cases, arising from the frequent failure of ϕ_4 and F_4'. The former contains a plane nodal curve, which invalidates it for most plane-problems, and the latter, which was particularly useful in higher space, now possesses three concurrent nodal lines which spoil it for line-problems as well. Hence it is often essential to adopt a correspondence method when surfaces of order higher than four are not available as special cases.

4. *The double curve.* We shall denote the double curve on F by Γ and its order by b, in accordance with the usual practice. It follows from § 1 that b is given by the equation

$$b=\int nn'=\binom{n}{2}-\tfrac{1}{2}a \quad \text{.....(1),}$$

the linear terms being found from special cases.

4·1. The number t of triple points on Γ (which are triple points of both Γ and F) is given by

$$t=\int bn'+b'n=\binom{n}{3}-\tfrac{1}{2}an+\tfrac{1}{3}(2a+2j+m) \quad \text{.....(2).}$$

4·2. To find the rank q of Γ, we replace F by F and F' in general position, so that we have for the rank of their intersection

$$\Delta q=nn'(n+n'-2)-2bn'-2b'n=an'+a'n.$$

Thus $$q=an-a-\tfrac{5}{2}j-m \quad \text{.....(3).}$$

This agrees with the corresponding scroll formula and vanishes identically for f_n.

4·3. The class of immersion ρ (the class of the developable of tangent planes to F at the points of Γ) is evidently the number of

* Basset, p. 31.

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intersections, with the double curve, of the first polar of an arbitrary point. Hence, since ρ is an additive character,

$$\Delta\rho = n'n(n-1) - 2bn' + nn'(n'-1) - 2b'n = an' + a'n,$$

and $$\rho = an - a - 2j - m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

agreeing with the scroll formula and vanishing identically for f_n.

4·4. The number h of apparent nodes is then given by

$$h = \binom{b}{2} - \tfrac{1}{2}q - 3t,$$

so that

$$h = 3\binom{n}{4} - \tfrac{1}{2}a\binom{n}{2} + an + \tfrac{1}{8}a^2 - \tfrac{1}{4}(5a + 3j + 2m) \quad \ldots(5).$$

For future purposes it is important to note that this is also obtained as the solution of

$$\Delta h = \tfrac{1}{2}nn'(n-1)(n'-1) + bb' + n(n'-2)b' + n'(n-2)b.$$

4·5. The genus p is given by

$$p = \binom{b-1}{2} - h - 3t = \tfrac{1}{2}q - b + 1$$

$$= \tfrac{1}{2}an - \binom{n}{2} - \tfrac{5}{4}j - \tfrac{1}{2}m + 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

4·6. The class

$$s = 3b(b-2) - 6(h+3t)$$

$$= 3\left\{an - \frac{a}{2} - \tfrac{5}{2}j - m - \binom{n}{2}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

This is obtained as the solution of the equation

$$\Delta s = 3nn'(n + n' - 3) - 6(nb' + n'b).$$

Formulae for multiple tangents.

5. *The bitangents.* These form a congruence of order

$$\mu_{2-2} = \int aa' = \tfrac{1}{2}a^2 + a - 3j - 2m \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

(agreeing with the scroll formula and f_n) and class

$$\nu_{2-2} = \tfrac{1}{2}a^2 + 4n - 5a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

5·1. The number of tangents at a point of F which are elsewhere tangent is

$$t_2 = a - 6 + 2\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where, as in previous work, ϵ is unity for scrolls and zero for surfaces.

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5·2. It follows at once from specialisation arguments that the bitangents which meet an arbitrary line generate a scroll of order

$$\rho_{2-2} = \mu_{2-2} + \nu_{2-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

whose curve of contact with F is of order

$$m_{2-2} = 2\nu_{2-2} + nt_2 = a^2 + an - 10a + 2(1+\epsilon)n \quad \ldots (12).$$

Thus the curve of contact with F of bitangents which belong to an arbitrary complex of order N is of order Nm_{2-2}.

5·3. To find the number of lines through a point of F which are elsewhere bitangent, we note that the tangent cone from that point is of order $a-2$, so that the required number is

$$t_{2-2} = \int^* \mu'_{2-2} + a'(a-2) = \mu_{2-2} - 2(a-6) - 4\epsilon \quad \ldots (13).$$

This vanishes for $f_3, f_4, F_4, F_4', \phi_4$.

The number of lines through a point of Γ which are elsewhere bitangent is

$$\delta_{2-2} = \int^* \mu'_{2-2} + a'(a-4) = \mu_{2-2} - 4(a-7) - 8\epsilon \quad \ldots (14).$$

This vanishes for ϕ_4, F_5, ϕ_5; it gives $-4 = d^{(4)}$ for F_4 and $d^{(4)}$ for the scroll (4, 8); and it gives $4 = d^{(5)}$ for F_5' and $d^{(5)}$ for F_5''.

Finally, the number of lines through a triple point of F which are elsewhere bitangent is

$$\tau_{2-2} = \int^* \mu'_{2-2} + a'(a-6) = \mu_{2-2} - 6(a-8) - 12\epsilon \quad \ldots (15),$$

vanishing† for ϕ_5 and J_6.

6. *The tritangents.* There is a scroll of tritangents whose order is given by

$$\Delta\mu_{2-2-2} = a'\mu_{2-2} + a'\nu_{2-2} + a\mu'_{2-2} + a\nu'_{2-2},$$

whence

$$\mu_{2-2-2} = 2\binom{a}{3} - 6\binom{a}{2} - a(3j + 2m - 4n) - 16n + 8\epsilon n + \tfrac{1}{3}(128j - 91a + 88m) \quad \ldots\ldots\ldots\ldots\ldots (16).$$

This vanishes for $f_3, f_4, F_4, F_4', \phi_4$.

6·1. To find the order m_{2-2-2} of the curve of contact we use § 5·2 and the fact that the curve of contact of tangents to F which belong to an arbitrary congruence (α, β) is of order $a\alpha + n\beta$; for the tangents to F whose points of contact lie in a given plane form

† It does not vanish for F_4' on account of the nodal lines through the triple point.

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a congruence (a, n), whence the result follows by Halphen's theorem. Thus

$$\Delta m_{2-2-2} = a' m_{2-2} + a' \mu_{2-2} + n' \nu_{2-2} \ldots,$$

where the terms omitted can be written down by symmetry. We obtain

$$m_{2-2-2} = 3\binom{a}{3} + n\binom{a}{2} - 15\binom{a}{2} - a(3j + 2m + \tfrac{5}{2}n - 2\epsilon n) + 4n^2 + \tfrac{239}{2}a - 112n + 4\epsilon(9n - 10a) \quad \ldots\ldots\ldots\ldots\ldots(17).$$

This formula agrees with f_n and gives the appropriate value of kl for ϕ_4 and ϕ_5, where $k = 4$ is determined from f_3. For (4, 8) and F_4 we have $-16 = 2d^{(4)}$ and $56 = kl + d^{(4)}$, respectively. F_5' gives $8 = d^{(5)}$ and F_5'' gives $kl + 2d^{(5)}$.

7. *The quadritangents.* The number of quadritangents is given by

$$\Delta \mu_{2-2-2-2} = a' \mu_{2-2-2} + \mu_{2-2} \mu'_{2-2} + \nu_{2-2} \nu'_{2-2} + a \mu'_{2-2-2},$$

whence

$$\mu_{2-2-2-2} = 2\binom{a}{4} - 9\binom{a}{3} + \binom{a}{2}\{4n - 3j - 2m\} - \tfrac{229}{12}a^2 + \tfrac{9}{2}j^2 + a\{\tfrac{229}{6}j + \tfrac{79}{3}m - 34n + 8\epsilon n\} + 6jm + 4\binom{m}{2} + 16\binom{n}{2} + \tfrac{2295}{2}a - 662j - 466m - 480n + 80\epsilon(12n - 13a) \quad \ldots\ldots\ldots\ldots(18).$$

This agrees with f_n, from which we have $k = 16$; from (4, 8) we obtain $d^{(4)} = -16$.

The formula also gives kl for ϕ_4, ϕ_5.

8. *The inflexional tangents.* These form a congruence of order

$$\mu_3 = 2j - a + m \quad \ldots\ldots\ldots\ldots\ldots\ldots(19),$$

agreeing with the scroll formula and f_n, and class

$$\nu_3 = 3(a - n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(20).$$

At each ordinary point of F there are $2 - \epsilon$ inflexional tangents and at a point of Γ there are $6 - 4\epsilon$ such tangents.

8·1. The numbers of lines through an ordinary point, a double point and a triple point of F, which are elsewhere inflexionally tangent, are as follows:

$$t_3 = \int^* \mu_3' = \mu_3 - 6 + 3\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

the constant being found from f_3.

$$\delta_3 = \int^* \mu_3' = \mu_3 - 12 + 6\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots(22).$$

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This vanishes for ϕ_4 and gives $6 = d^{(4)}$ for F_4 and (4, 8).

$$\tau_3 = \mu_3 - 18 + 9\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23),$$

vanishing for ϕ_5.

8·2. For future purposes we note that the inflexional tangents which meet an arbitrary line generate a scroll of order

$$\rho_3 = \mu_3 + \nu_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24),$$

whose curve of contact with F is of order

$$m_3 = \nu_3 + 2n - \epsilon n = 3a - (1+\epsilon)\, n \ldots\ldots\ldots\ldots(25).$$

The curve of contact of inflexional tangents which belong to a complex of order N is of order Nm_3.

9. There is a finite number of $3-3$-tangents given by

$$\begin{aligned}\mu_{3-3} &= \int \mu_3\mu_3' + \nu_3\nu_3' \\ &= \tfrac{1}{2}(2j - a + m)^2 + \tfrac{9}{2}(a-n)^2 + 327a - 168j \\ &\quad - 2\tfrac{13}{2}m - \tfrac{33}{2}n + 3\epsilon(11n - 21a) \quad \ldots\ldots\ldots(26).\end{aligned}$$

This agrees with f_n; from f_3 we obtain $k = 9$, whence the formula is found to give kl for ϕ_4, ϕ_5. From (4, 8) we have $d^{(4)} = -18$, and F_4 then gives $126 = 16k + d^{(4)}$. Similarly, for F_5', the formula gives $d^{(5)} = 36$ and, for F_5'', $189 = 13k + 2d^{(5)}$.

10. The scroll S_{3-2} of $3-2$-tangents is of order

$$\begin{aligned}\mu_{3-2} &= \int a'(\mu_3 + \nu_3) \ldots \\ &= 4\binom{a}{2} + a(2j + m - 3n) + 38a - 48j - 30m \\ &\quad + 18n - 12\epsilon n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(27).\end{aligned}$$

This agrees with f_n and vanishes for F_4, F_4', ϕ_4.

10·1. The curve $C_{3:}$ of 3-point contacts is of order

$$\begin{aligned}m_{3:} &= \int a' m_3 + a m_3' \\ &= 3a^2 - an - \epsilon an + 48(n-a) + 18\epsilon(a-n) \quad \ldots(28),\end{aligned}$$

agreeing with f_n, from which $k = -2$, whence we have the value kl for ϕ_4. The scroll (4, 8) gives $d^{(4)} = 4$ and F_4 gives $-28 = kl + d^{(4)}$.

10·2. The curve $C_{2:}$ of simple contacts is of order

$$\begin{aligned}m_{2:} &= \int a'\mu_3 + n'\nu_3 \ldots \\ &= a(2j - a + m + 3n) - 3n^2 + 48(n-a) + 18\epsilon(a-n) \\ &\qquad\qquad \ldots\ldots(29).\end{aligned}$$

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This agrees with the formula for f_n, which is misprinted in Salmon*. We obtain $k = -3$ and $d^{(4)} = 12$, which agree with the value found for F_4, namely $-36 = kl + d^{(4)}$.

For ϕ_5 the two curves, suitably counted, should form the complete intersection of ϕ_5 and S_{3-2}. The calculated values of μ_{3-2}, $m_{2:}$ and $m_{3:}$ are 36, $15 = 45 - 3.10$, and $10 = 30 - 2.10$, whence $m_{2:} = 45$, $m_{3:} = 30$; these numbers satisfy

$$5\mu_{3-2} = 3m_{3:} + 2m_{2:} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

As a check for F_5' and F_5'', we can obtain a verification of this last equation, without knowing the nodal line constants $d_{2:}$ and $d_{3:}$ for the two problems. The calculated values for F_5' are $\mu_{3-2} = 360$, $m_{3:} = 258$, $m_{2:} = 489$, and for F_5'', $\mu_{3-2} = 270$, $m_{3:} = 160$, $m_{2:} = 309$. The actual values in the two cases are therefore $m_{3:} = 258 - d_{3:}$, $m_{2:} = 489 - d_{2:}$ for F_5' and

$$m_{3:} = 160 + 2.13 - 2d_{3:}, \quad m_{2:} = 309 + 3.13 - 2d_{2:}$$

for F_5''. In each case (i) leads to the result $3d_{3:} + 2d_{2:} = -48$.

11. The number of $3-2-2$-tangents is given by

$$\Delta\mu_{3-2-2} = a'\mu_{3-2} + \mu_{2-2}\mu_3' + \nu_{2-2}\nu_3' \ldots,$$

whence

$$\begin{aligned}\mu_{3-2-2} = 6\binom{a}{3} + \binom{a}{2}\{2j - 3n + m\} + a(23a - 42j - \tfrac{53}{2}m \\ + \tfrac{87}{2}n - 12\epsilon n) - 6j^2 - 2m^2 - 7jm - 12n^2 \\ - 1970a + 1068j + 720m + 912n + 24\epsilon(15a - 8n) \ldots(30).\end{aligned}$$

This agrees with f_n (f_3 giving $k = -36$), whence we have the usual verifications kl for ϕ_4, F_5, ϕ_5. From (4, 8) we obtain $d^{(4)} = 48$, whence follows another verification for F_4.

12. *The flecnodal tangents.* The flecnodal or 4-point tangents form a scroll S_4 whose order $\mu_4 = Aa + Jj + Mm + Nn$, since $\Delta\mu_4 = 0$.

We have $$\mu_4 = 11j - 10a + 6m - 4n + 4\epsilon n \quad \ldots\ldots\ldots\ldots\text{(31)},$$

agreeing with the formulae for f_n and the scroll (save for the one usual discrepancy).

12·1. The order m_4 of the curve of contact is found as follows†. Consider the scroll S of inflexional tangents at points of a plane section π; this is of order‡ $3a - n$, and meets F in the curve π counted six times, so that its residual intersection is of order

* Salmon, p. 287; the coefficient -12 should be $+12$.

† This method, which is used by James, seems to have originated with Voss, *Math. Annalen* 9 (1876), 483.

‡ In this argument, all ϵ-considerations have been omitted in order to save unnecessary complexity.

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$n(3a-n)-6n$. This meets the plane of section in the points m_4, the points of Γ counted β times, say, and the intersections of the curve with the inflexional tangents lying in π, $(n-3)\,\nu_3$ in number. Thus we have

$$n(3a-n-6) = m_4 + \beta b + 3(a-n)(n-3).$$

Now if m_4 is to be linear in the characters we must have $\beta = 4$, whence

$$m_4 = 11a - 13n + \epsilon(6n-5a) \quad \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

From f_3 we have $k=1$ (a known theoretical result). For ϕ_4 the flecnodal curve should form the complete intersection of ϕ_4 and S_4. Now $\mu_4 = 20$ and m_4 gives $36 = 20 + kl$, in accord with the prediction.

From (4, 8) we have $d^{(4)} = 10$. Then for F_4 we obtain $\mu_4 = 48$, $m_4 = 58$. Hence it appears that for this surface the 16 simple lines form part of the flecnodal curve.

It may be added that the formula for m_4 agrees with Cayley's theorem* that for a surface possessing double and cuspidal curves of orders b and c respectively,

$$m_4 = n(11n-24) - 22b - 27c.$$

13. There is a finite number of $4-2$-tangents given by

$$\mu_{4-2} = \int a\mu_4' + a'\mu_4 = a\mu_4 + 656a - 328j - 208m - 336n + 8\epsilon(10n-17a) \quad \ldots\ldots\ldots\ldots\ldots\ldots (33).$$

This agrees with f_n and gives kl for ϕ_4, ϕ_5, F_5 $(k=16)$. From (4, 8) we have $d^{(4)} = -32$, whence the check for F_4 is $224 = 16k + d^{(4)}$. For F_5' we have $48 = d^{(5)}$ and for F_5'', $\mu_{4-2} = 304 = 13k + 2d^{(5)}$.

14. *The 5-point tangents.* There is a finite number μ_5 of 5-point tangents. Since $\Delta\mu_5 = 0$ we obtain

$$\mu_5 = -130a + 60j + 35m + 75n + 10\epsilon(3a-2n) \ldots (34),$$

agreeing with f_n and ϕ_4 $(k=-5)$. From (4, 8) we have $d^{(4)} = 20$ and for F_4, $\mu_5 = -60 = 16k + d^{(4)}$.

15. *Lines through a triple point.* Besides the τ_{2-2} and τ_3 formulae there are three results to be given; they may be obtained by considering the cone of order $b-3$, which contains Γ and has its vertex at a triple point T, and the tangent cone from the same point, of order $a-6$.

* Basset, p. 40.

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Since the former cone is of genus p, we have for the number of lines through T which meet Γ twice elsewhere,

$$\tau_{1;1} = \tfrac{1}{2}(b-4)(b-5) - p - 3(t-1)$$
$$= 3\binom{n}{4} - 3\binom{n}{2} + an - \tfrac{1}{2}a\binom{n}{2} + \tfrac{1}{8}a^2 - \tfrac{3}{4}j + \tfrac{1}{4}a - \tfrac{1}{2}m + 12 - 30\epsilon \quad \ldots\ldots(35).$$

The number of lines through T which are elsewhere secant to Γ and tangent to F is

$$\tau_{1;2} = \tfrac{1}{2}a^2n - \tfrac{5}{2}an - \tfrac{1}{2}a^2 - 3n^2 + \tfrac{19}{2}a - \tfrac{3}{4}j + \tfrac{1}{2}m + 3n + 15 + 3\epsilon \quad \ldots\ldots(36).$$

The number of lines through T which are elsewhere tangent in a point of Γ is

$$\tau_{2;} = \int^{*} \rho' + a'n + (a-6)n' = \rho - 6n + 18 + 3\epsilon(n-6) \quad \ldots\ldots(37).$$

The constant is found from the fact that $\tau_{2;}$ must vanish for ϕ_5.

Secants of the double curve.

16. We have now to consider systems of lines which are secant to Γ or tangent to F in a point of Γ. The majority of the formulae are obtained by correspondence methods due in the first place to Schubert and extensively developed by James. These depend on the following theorem, which we use later in its dual form:

"If the points of two curves—which may be identical—of orders a and b respectively are in (x, x') correspondence, then the number of united points is $xb + x'a - c$, where c is the order of the scroll of lines joining corresponding points, counted twice in involutory cases."

In the subsequent sections there is an essential difference between the nature of the results for scrolls and for surfaces, and in order to avoid confusion it has been found advisable to omit the ϵ-terms altogether; the corresponding formulae may be referred to in James's work. The suffix notation used is due to James.

17. There is a congruence of secants of Γ which are elsewhere tangent; evidently the class is $\nu_{1;2} = (a-6)b$. The order is

$$\mu_{1;2} = \int ab' + a'b + (nn' - 2n'a) + (nn' - 2na')$$
$$= a\left\{\binom{n}{2} - \tfrac{1}{2}a\right\} - 2an + 2a + 3j + 2m \quad \ldots\ldots(38),$$

agreeing with the scroll formula and vanishing identically for f_n.

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The number of such lines passing through a point of F is

$$t_{1;2} = \int^{*} (nn'a - 2n'a) + \{nn'(a-2) - 2n'a\} + \mu'_{1;2} + a'b + (a-2)b'$$
$$= \mu_{1;2} - 2b \quad \ldots\ldots\ldots\

Γ is sextuple on S_3; and the corresponding sheets have inflexional contact along it, so that it counts 24 times* in the intersection of S_3; and F. The residual curve is thus of order

$$m'_{3;} = n\mu_{3;} - 24b \quad \text{.....(44).}$$

20. There are two scrolls whose properties are deduced from a like process: the scroll $S_{2;\,2}$ of tangents at points of Γ which are elsewhere tangent, and $S_{1;\,2-2}$, generated by secants of Γ which are elsewhere bitangent. Consider the scroll Σ of bitangents which meet an arbitrary line l; this is of order ρ_{2-2} and touches F along a curve of order m_{2-2} (§ 5·2), so that the residual intersection is of order $n\rho_{2-2} - 2m_{2-2}$; on these two curves the generators of Σ define a correspondence $(n-4, 2)$, whence we obtain

$$\mu_{3-2} + \mu_{2;\,2} = (n-4)\, m_{2-2} + 2\,(n\rho_{2-2} - 2m_{2-2}) - 2\,(n-4)\,\rho_{2-2},$$

or

$$\mu_{2;\,2} = (n-8)\, m_{2-2} + 8\rho_{2-2} - \mu_{3-2} \quad \text{.....(45).}$$

20·1. The curve of contact is found by means of the scroll Σ' of bitangents, one of whose contacts is on a given plane π. This scroll meets F in the curve π counted $a-6$ times, and in the ν_{2-2} bitangents in π, counted twice. Hence it is of order $(a-6)\,n + 2\nu_{2-2}$. The other curve of contacts is of order $a^2 + n^2 - 6n + \epsilon\,(6n - 5a)$, so that the residual intersection is of order

$$n_{2-2} = n\,\{(a-6)\,n + 2\nu_{2-2}\} - 2n\,(a-6) - 2\,(a^2 + n^2 - 6n),$$

whence

$$n_{2-2} = an^2 + a^2n - 2a^2 - 12an + 24n \quad \text{.....(46).}$$

This vanishes for f_4 and gives kl $(k=-2)$ for f_3 and ϕ_4. From the scroll formula we find that $d^{(4)} = 8$ and hence obtain another verification from F_4.

20·2. From the correspondence between the points on the curve of contact and on the residual intersection we have

$$m_{2;\,2} + m_{2:} = (n-4)\,(a^2 + n^2 - 6n) + n_{2-2} - (n-4)\,\{(a-6)\,n + 2\nu_{2-2}\}$$
$$\text{.....(47).}$$

The section of F by either tangent plane at a point of Γ is a curve with an ordinary triple point at the point of contact, so that there are $a-12$ lines in the plane which are elsewhere tangent to F. Thus Γ is a $2(a-8)$-ple curve on $S_{2;\,2}$ such that each set of $a-12$ sheets has a common tangent plane at each point of Γ. Thus the residual intersection is of order

$$m'_{2;\,2} = n\mu_{2;\,2} - 2m_{2;\,2} - 6\,(a-8)\,b \quad \text{.....(48).}$$

* For a scroll the multiplicity is only double and the curve counts eight times.

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21. For the scroll $S_{1;\,2-2}$ we again consider Σ and the correspondence $(n-5, n-5)$ set up by its generators on the residual intersection of Σ and F, of order $n\rho_{2-2} - 2m_{2-2}$. Hence

$$3\mu_{2-2-2} + 2\mu_{1;\,2-2} = 2(n-5)(n\rho_{2-2} - 2m_{2-2}) - (n-4)(n-5)\rho_{2-2} \quad \ldots\ldots(49).$$

The multiplicity of Γ on this scroll is δ_{2-2} (§ 5·3). Then if we consider the correspondence $(n-5, n-5)$ on the curve of order n_{2-2} on Σ' (§ 20·1), we obtain the order of the curve of contact of $S_{1;\,2-2}$, given by

$$2m_{2-2-2} + 2m_{1;\,2-2} = 2(n-5)n_{2-2} - (an - 6n + 2\nu_{2-2}) \quad \ldots\ldots(50).$$

The residual intersection is therefore of order

$$m'_{1;\,2-2} = n\mu_{1;\,2-2} - 2\delta_{2-2}b - 2m_{1;\,2-2} \quad \ldots\ldots\ldots(51).$$

22. *Secants satisfying four conditions.* From the various intersections with F of the scrolls S_{2-2-2}, S_{3-2} and S_4 we obtain nine formulae for tangents of F which meet Γ in one point and satisfy four conditions in all.

22·1. *The scroll* S_{2-2-2}.

(i) Consider the (2, 2) correspondence set up by the generators on the curve of contact, which is of order m_{2-2-2}. The united points are due to the $4-2$-tangents and the δ_2 tangents of Γ which are elsewhere tangent (counted twice). Thus we have

$$\mu_{4-2} + 2\delta_2 = 4m_{2-2} - 6\mu_{2-2-2} \quad \ldots\ldots\ldots\ldots(52).$$

(ii) The correspondence $(n-6, 3)$ between the curve of contact and the residual intersection, of order $m'_{2-2-2} = n\mu_{2-2-2} - 2m_{2-2-2}$, has united points arising from the $3-2-2$-tangents and the tangents in points of Γ which are elsewhere bitangent. Thus

$$\mu_{3-2-2} + \mu_{2;\,2-2} = 3m'_{2-2-2} + (n-6)m_{2-2-2} - 3(n-6)\mu_{2-2-2},$$

or

$$\mu_{2;\,2-2} = 18\mu_{2-2-2} + (n-12)m_{2-2-2} - \mu_{3-2-2} \quad \ldots\ldots(53).$$

(iii) The correspondence $(n-7, n-7)$ on the residual curve has coincidences arising from the quadritangents, counted four times, and the secants to Γ which are elsewhere tritangent. Hence

$$4\mu_{2-2-2-2} + 2\mu_{1;\,2-2-2} = 2(n-7)m'_{2-2-2} - (n-6)(n-7)\mu_{2-2-2},$$

or

$$\mu_{1;\,2-2} = \tfrac{1}{2}(n+6)(n-7)\mu_{2-2-2} - 2(n-7)m_{2-2-2} - 2\mu_{2-2-2-2} \quad \ldots\ldots(54).$$

22·2. *The scroll* S_{3-2}.

(i) The correspondence between the curves of 3-point and 2-point contact has coincidences due to the 5-point tangents and the tangents δ_5 to Γ which have 5-point intersection with F. Thus

$$\mu_5 + \delta_5 = m_{3:} + m_{2:} - \mu_{3-2} \quad \ldots\ldots\ldots\ldots(55).$$

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(ii) The correspondence $(n-5, 1)$ between $C_{3:}$ and the residual intersection, of order $m' = n\mu_{3-2} - 3m_{3:} - 2m_{2:}$, has united points which arise from the $4-2$-tangents and the lines which are inflexionally tangent at points of Γ and which are elsewhere tangent. Hence

$$\mu_{4-2} + \mu_{3;2} = m' + (n-5)\, m_{3:} - (n-5)\, \mu_{3-2},$$

so that

$$\mu_{3;2} = 5\mu_{3-2} + (n-8)\, m_{3:} - 2m_{2:} - \mu_{4-2} \quad \ldots\ldots(56).$$

(iii) Similarly, the correspondence between $C_{2:}$ and the curve m' gives the tangents at points of Γ which are elsewhere inflexionally tangent, in number

$$\mu_{2;3} = 5\mu_{3-2} - 3m_{3:} + (n-7)\, m_{2:} - 2\mu_{3-3} \quad \ldots\ldots(57).$$

(iv) From the correspondence $(n-6, n-6)$ on the curve m' we obtain the secants to Γ which are elsewhere $3-2$-tangent; thus

$$2\mu_{3-2-2} + 2\mu_{1;3-2} = 2(n-6)\, m' - (n-5)(n-6)\, \mu_{3-2},$$

which gives

$$\mu_{1;3-2} = \tfrac{1}{2}(n+5)(n-6)\, \mu_{3-2} - (n-6)(2m_{2:} + 3m_{3:}) - \mu_{3-2-2} \quad \ldots\ldots(58).$$

22·3. *The scroll S_4.* There are two remaining formulae:

(i) From the correspondence $(n-4, 1)$ between the curve of contact and the residual intersection, of order $n\mu_4 - 4m_4$, we obtain the lines which have 4-point contact at a point of Γ, namely

$$\mu_{4;} = 4\mu_4 + (n-8)\, m_4 - \mu_5 \quad \ldots\ldots\ldots\ldots\ldots(59).$$

(ii) The correspondence $(n-5, n-5)$ on the residual intersection gives the secants to Γ which are elsewhere 4-point tangent,

$$\mu_{1;4} = \tfrac{1}{2}(n+4)(n-5)\, \mu_4 - 4(n-5)\, m_4 - \tfrac{1}{2}\mu_{4-2} \ldots(60).$$

Chords of the double curve.

23. *The scroll $S_{1;1;2}$.* This is the scroll of chords of Γ which are elsewhere tangent to F. To obtain its properties we require the following elementary results:

(i) The scroll of chords of Γ which meet an arbitrary line l is of order $\mu_{1;1;} = h + \tfrac{1}{2}b(b-1)$ and contains Γ as a multiple curve of order $b-1$. Hence

(ii) It meets F in a residual curve C of order

$$m_{1;1;} = n\,\{h + \tfrac{1}{2}b(b-1)\} - 2b(b-1).$$

(iii) The trisecants of Γ generate a scroll $S_{1;1;1;}$ of order

$$\mu_{1;1;1;} = (b-2)\,\{h - \tfrac{1}{6}b(b-1)\} + t \quad \ldots\ldots\ldots(61).$$

This is an extension, due to Schaake*, of the well-known formula.

* Schaake, *Proc. Roy. Soc. Amsterdam*, 27 (1924), 362—370.

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The multiplicity of Γ on this scroll is $\delta_{1;1;1;} = h - (b-2)$, so that the residual intersection is of order

$$m'_{1;1;1;} = n\mu_{1;1;1;} - 2b\delta_{1;1;1;}.$$

23·1. The order of $S_{1;1;2}$ is obtained from the correspondence $(n-5, n-5)$ between points of C which lie on the same secant of l. Thus

$$\mu_{1;1;2} + 6\mu_{1;1;1;} = 2(n-5)m_{1;1;} - (n-4)(n-5)\mu_{1;1;} \quad \ldots(62).$$

To find the multiplicity of Γ on this scroll, we consider the cone of order $b-1$ whose vertex is at a point of Γ; this meets F again in a curve of order $n(b-1)-2b$, on which its generators determine a correspondence leading to the equation

$$\delta_{1;1;2} + 4\delta_{1;1;1;} = 2(n-5)(nb-n-2b) - (n-4)(n-5)(b-1) \quad \ldots\ldots(63).$$

23·2. For the curve of contact, consider the scroll S of secants to Γ which are tangent to F in points of a plane section π. Its section by π consists of π counted b times, and $(a-6)b$ lines in the plane, so that its order is $b(a+n-6)$. Its nb sheets along the curve π are tangent to S, and Γ is $(a-4)$-ple on S; the residual intersection is therefore of order

$$m' = nb(a+n-6) - 2bn^2 - 2b(a-4).$$

The correspondence set up on this curve by the generators of S gives

$$4m_{1;1;2} + m_{1;2-2} = 2(n-5)m' - (n-4)(n-5)b(a+n-6) \quad \ldots\ldots(64).$$

Since F and $S_{1;1;2}$ meet in Γ and touch along the curve of contact, their residual intersection is of order

$$m'_{1;1;2} = n\mu_{1;1;2} - 2b - 2m_{1;1;2} \quad \ldots\ldots\ldots\ldots(65).$$

24. *The scroll* $S_{1;2;}$. Finally we have the scroll of tangents to F in a point of Γ which are elsewhere secant to Γ. Consider the scroll S' generated by tangents to F in points of Γ, which meet a given line l; its order is $\rho + 2b$, by §4·3, and it contains Γ as a double curve*, so that its residual intersection with F is of order

$$m'_{2;} = n(\rho + 2b) - 6b.$$

The generators of S' give a correspondence on this curve, from which we obtain

$$\begin{aligned} 2\mu_{1;2;} + \mu_{2;2} &= 2(n-3)m'_{2;} - (n-2)(n-3)(\rho+2b) \\ &= (n-3)(n+2)(an+n^2-a-2j-m-n) - 12(n-3)b \\ &\qquad \ldots\ldots(66). \end{aligned}$$

* Its tangent planes along Γ are the tangent planes to F.

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24·1. This scroll has two sets of sheets respectively tangent to the sheets of F along Γ; each set contains $b-1$ sheets. There is another set of sheets along Γ, due to secants which are elsewhere tangent to F in points of Γ; evidently the number is

$$\delta_{1;2;} = \int^{*} \rho' + a'n + (a-4)\,n' = \rho - 4n + 12 \quad \ldots(67),$$

the constant being found from ϕ_4, for which the result must vanish. As a verification, note that from the case F_4 we have $d^{(4)} = -2$, and that the same value is obtained from the scroll formula and (4, 8).

25. *Chords satisfying four conditions.* There are six formulae of this type which are obtained as follows:

(i) The correspondence between the curve of contact and the residual intersection on $S_{1;3}$ gives the chords of Γ which are inflexionally tangent at one extremity: thus

$$\mu_{1;3;} + \mu_{1;4} = m'_{1;3} + (n-5)\,m_{1;3} - (n-5)\,\mu_{1;3} \quad \ldots(68).$$

(ii) The correspondence between points of the residual intersection gives the chords of Γ which are elsewhere inflexionally tangent:

$$\mu_{1;2-3} + 4\mu_{1;1;3} = 2\,(n-6)\,m'_{1;3} - (n-5)\,(n-6)\,\mu_{1;3} \quad \ldots(69).$$

(iii) The correspondence between the curves of contact and residual intersection on $S_{2;2}$ gives the chords of Γ which are tangent to F at both extremities:

$$\mu_{2;3} + 2\mu_{2;2;} = m'_{2;2} + (n-5)\,m_{2;2} - (n-5)\,\mu_{2;2} \quad \ldots(70).$$

(iv) The correspondence on the curve of contact of $S_{1;2-2}$ gives the tangents to Γ which are elsewhere secant†:

$$\mu_{1;4} + 2\delta_{1;} = 2m_{1;2-2} - 2\mu_{1;2-2} \quad \ldots\ldots\ldots\ldots(71).$$

(v) From the correspondence between the curves of contact and residual intersection, we obtain the chords which are tangent to F at one extremity and which are elsewhere tangent:

$$\mu_{1;2-3} + \mu_{1;2;2} = m'_{1;2-2} + (n-6)\,m_{1;2-2} - (n-6)\,\mu_{1;2-2} \quad \ldots(72).$$

(vi) The correspondence on the curve of residual intersection gives the chords of Γ which are elsewhere bitangent:

$$3\mu_{1;2-2-2} + 4\mu_{1;1;2-2} = 2\,(n-7)\,m'_{1;2-2} - (n-6)\,(n-7)\,\mu_{1;2-2} \quad \ldots\ldots(73).$$

† The correspondence given by James (J. p. 223) is misprinted; the coefficient of Z should be 2.

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26. *Trisecants of the double curve.* There are three formulae for trisecants which satisfy one further condition:

(i) The correspondence between the curves of contact and residual intersection on $S_{1;1;2}$ gives the trisecants which are tangent at one intersection:

$$\mu_{1;1;3} + \mu_{1;1;2} = m'_{1;1;2} + (n-6)\, m_{1;1;2} - (n-6)\, \mu_{1;1;2} \quad \ldots(74).$$

(ii) The correspondence on the residual intersection gives the trisecants which are elsewhere tangent to F:

$$2\mu_{1;1;2-2} + 6\mu_{1;1;1;2} = 2\,(n-7)\, m'_{1;1;2} - (n-6)\,(n-7)\, \mu_{1;1;2} \quad \ldots\ldots(75).$$

(iii) The correspondence on the residual intersection of $S_{1;1;1;}$ gives the quadrisecants of Γ:

$$8\mu_{1;1;1;1} + \mu_{1;1;1;2} = 2\,(n-7)\, m'_{1;1;1;} - (n-6)\,(n-7)\, \mu_{1;1;1;} \quad \ldots\ldots(76).$$

II.

27. Since a plane can be made to satisfy three given conditions, it follows that there are three developables associated with any surface F:

(i) The bitangential developable **B**, whose tangent planes touch F at two distinct points;

(ii) The spinodal developable **C**, whose tangent planes have cuspidal contact with F;

(iii) The flecnodal developable **F**, whose tangent planes touch F at a flecnode.

Each of these developables has an edge of regression and a curve of contact with F, which are briefly considered in the sequel. In addition, their tangent planes may be made to satisfy one other condition, giving three types of tangent planes as follows:

(i) planes ϖ_1 having stationary and ordinary contact with F;

(ii) planes ϖ_2 having flecnodal and ordinary contact;

(iii) tritangent planes ϖ_3.

Finally, there are a finite number of planes ϖ_4 whose point of contact is a biflecnode, of planes ϖ_5 which have stationary contact with **C**, and of planes ϖ_6, whose point of contact is a hyperflecnode.

27·1. In this part of the work, no systematic attempt has been made to obtain the ϵ-terms for the corresponding scroll formulae; the results for **B** must be identical with those given for Γ, in virtue of the self-duality of the scroll; those for **C** should vanish identically since a scroll has no cuspidal curve; and those for **F** have already been considered.

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27·2. The principle of redundancy applies to developables of contact in the same way as it does to curves:—a line on F gives a pencil of planes which counts k times in the developable considered, where k is a constant for the problem. Also the nodal line constant $d^{(4)}$ must be the same as for the dual problem, since it is determined originally from a scroll and the scroll reciprocates into itself. The same considerations as in § 2·1 apply to the absence of certain linear terms in various solutions.

28. *Notation.* The notation for the characters of Γ, $\mathbf{B}$ and $\mathbf{C}$ is, in the main, that of Cayley and Zeuthen, except that h and k are interchanged and accented symbols are replaced by Clarendon letters. For the six singular tangent planes, the notation is that of Salmon and Basset. The flecnodal developable is not considered by Cayley or Zeuthen; the suffixes used are the same as in § 12. The complete list is as follows:

The double curve.

b, the order; q, the rank; s, the class;
h, the number of apparent nodes;
t, the number of triple points;
j, the number of pinch-points;
ρ, the class of immersion.

The bitangential developable.

$\mathbf{b}$, the class; $\mathbf{q}$, the order; $\mathbf{s}$, the order of the edge of regression;
$\mathbf{h}$, the number of apparent double planes;
ϖ_3, the number of tritangent planes;
ϖ_1, the number of stationary tangent planes;
$\boldsymbol{\rho}$, the order of the bitangential curve.

The spinodal developable.

$\mathbf{c}$, the class; $\mathbf{r}$, the order; $\mathbf{m}$, the order of the edge of regression;
$\mathbf{k}$, the number of apparent double planes;
ϖ_5, the number of stationary tangent planes;
$\boldsymbol{\sigma}$, the order of the spinodal curve.

The flecnodal developable.

ν_4, the class; μ_4, the order;
ϖ_6, the number of hyperflecnodal tangent planes;
ϖ_2, the number of flecnodal tangent planes which are elsewhere tangent;
ϖ_4, the number of biflecnodal tangent planes;
m_4, the order of the flecnodal curve.

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29. *Some preliminary results.* We first obtain formulae which are the dual of those given in §§ 5·2, 8·2; and, as in the sequel, denote dual results by the same symbols in Clarendon type.

(i) To find the class of the developable of contact of tangents to F which belong to an arbitrary congruence (α, β).

By this is meant the developable of tangent planes to F at the points of contact. We proceed by the dual of the ordinary method. Suppose that the congruence consists of the lines which pass through α points or which lie in β planes. The number of tangent planes through a given point P is found by joining it to each of these points and, secondly, supposing it situated in one of the planes. We thus obtain for the class of the developable the number $m\alpha + a\beta$.

(ii) The developable of contact of bitangents meeting a given line l is of class

$$\mathbf{m}_{2-2} = mt_2 + 2\mu_{2-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(77).$$

This follows at once by supposing the point P to lie on l.

(iii) Hence the developable of contact of bitangents which belong to a complex of order N is of class $N\mathbf{m}_{2-2}$.

(iv) The developable of contact of inflexional tangents meeting a given line l is of class

$$\mathbf{m}_3 = \mu_3 + (2 - \epsilon)\, m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(78).$$

This is obtained as in (ii). We also have a result similar to (iii).

The Spinodal Developable.

30. It follows at once from the nature of $\mathbf{C}$ that all additive characters connected with it are linear functions of the characters of F. Also, since a scroll is self-dual and does not possess a cuspidal curve, it cannot have a developable $\mathbf{C}$. Some of the formulae obtained do not, however, vanish for scrolls, on account of the singular generators, which occur redundantly in these results.

30·1. The class of $\mathbf{C}$ is given by

$$\mathbf{c} = 4m - 4a + 2j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(79).$$

This vanishes for f_1, f_2, gives the correct value* 24 for f_3, and vanishes for all scrolls. It also vanishes for F_4'.

30·2. The order of the edge of regression is

$$\mathbf{m} = 30j - 60a + 28m + 32n \quad \ldots\ldots\ldots\ldots\ldots(80),$$

agreeing with f_n and vanishing for scrolls.

* Salmon, p. 162.

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30·3. The number of apparent double planes is

$$\mathbf{k} = \int \mathbf{cc}' = 2(2m - 2a + j)^2 + 78a - 39j - 38m - 40n \ldots(81),$$

agreeing with f_n.

30·4. The order $\boldsymbol{\sigma}$ of the spinodal curve is found as follows. As with all other curves of contact, we may expect it to be of the form $Aa + Nn$; it must vanish for f_2 and give the value 12 for f_3*. Hence

$$\boldsymbol{\sigma} = 4(a - n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(82).$$

This agrees with Cayley's theorem† that, for a surface containing double and cuspidal curves of orders b and c respectively, the value of $\boldsymbol{\sigma}$ is $4n(n-2) - 8b - 11c$. It appears that, for a scroll, the singular generators counted twice form the spinodal curve.

31. *Intersections with the double curve.* We now establish an important principle which will be used again later. If a curve of order c on F is such that c is a *linear* function of the characters of F, and if F is replaced by surfaces F, F', then c consists of two portions on F and F' respectively. It is obvious that the number of intersections of c and Γ is given by

$$i = \int cn' + c'n.$$

Three constants of integration may be found from the fact that i must vanish for f_n; the coefficient of j is then determined from ϕ_4, for which Γ is a conic.

31·1 Thus the character $\mathbf{i}$ for the spinodal curve is

$$\mathbf{i} = \int \boldsymbol{\sigma} n' + \boldsymbol{\sigma}' n = 4(a - n)\, n - 4j - 4m + 4n \ldots\ldots(83),$$

vanishing for f_n and giving the value 16 for ϕ_4.

31·2. The character i_4 for the flecnodal curve is found from $\int m_4 n' + m_4' n$; the solution which vanishes for f_n and gives 20 for ϕ_4 (§ 12·1) is $(11a - 13n)\, n + 2a - 15j - 11m + 13n$; but this includes the pinch-points‡. Removing these, we have for the number of free intersections with Γ

$$i_4 = (11a - 13n)\, n + 2a - 16j - 11m + 13n \ldots\ldots(84).$$

* Salmon, p. 302.
† Basset, p. 39.
‡ Zeuthen, p. 469.

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32. In order to find the order $\mathbf{r}$ we use a modification of a method due essentially to Schubert. The curve of contact of inflexional tangents of F which meet a line l is of order m_3 (§ 8·2). Now $m_3 = (3n-4)\,n - 6b$. Hence this curve is the residual intersection with F of a surface Φ of order $3n-4$, containing Γ as a triple curve, and touching the spinodal curve at each intersection. At such a point the generator of $\mathbf{C}$ must meet l, so that we have

$$\mathbf{r} = \tfrac{1}{2}\{(3n-4)\,\boldsymbol{\sigma} - 3\mathbf{i}\} = 6j - 8a + 6m + 2n \quad \ldots\ldots(85).$$

33. *The tacnodal planes* $\boldsymbol{\varpi}_5$. To find these we use the elementary correspondence theorem, due to Chasles, for two systems of planes with a common axis. Through any line l draw a plane π meeting the spinodal curve in $\boldsymbol{\sigma}$ points P; at each of these points draw the cuspidal tangent to F, meeting F in $n-3$ further points P' which, joined to l, give planes π'. Since $\mathbf{C}$ meets F in a residual curve of order $n\mathbf{r} - 3\boldsymbol{\sigma}$, the correspondence between π, π' is $((n-3)\,\boldsymbol{\sigma}, n\mathbf{r} - 3\boldsymbol{\sigma})$. The united planes are given by the planes $\boldsymbol{\varpi}_5$, the lines PP' which meet l, $(n-3)\,\mathbf{r}$ in number, and the points $\mathbf{i}$. Hence we have

$$\boldsymbol{\varpi}_5 + (n-3)\,\mathbf{r} + \mathbf{i} = (n-3)\,\boldsymbol{\sigma} + n\mathbf{r} - 3\boldsymbol{\sigma}.$$

Simplifying we obtain

$$\boldsymbol{\varpi}_5 = 16j - 48a + 22m + 26n \quad \ldots\ldots\ldots\ldots\ldots(86),$$

agreeing with the known formula for f_n.

The Bitangential Developable.

34. The class

$$\mathbf{b} = \int mm' = \binom{m}{2} + \tfrac{11}{2}a - 3j - 6m \quad \ldots\ldots\ldots(87).$$

This agrees with f_n, reduces to b for a scroll, and vanishes for F_4'. For ϕ_4, $\mathbf{b} = 26 = 16 + 5\,.\,2$; this agrees with the known fact* that every plane drawn through one of the 16 lines is bitangent to the surface, and that the rest of $\mathbf{B}$ consists of the five Kummer's cones.

34·1. The order $\mathbf{q}$ is given, as in § 4·2, by

$$\Delta\mathbf{q} = mm'(m+m'-2) - 2m'\mathbf{b} - 2m\mathbf{b}' - 3m'\mathbf{c} - 3m\mathbf{c}' = am' + a'm.$$

This serves as a verification for $\mathbf{b}$ and $\mathbf{c}$, since for a scroll the non-linear term in $\mathbf{q}$ must be the same as in q.

We thus obtain

$$\mathbf{q} = am + 11a - \tfrac{17}{2}j - 11m - 2n \quad \ldots\ldots\ldots\ldots(88).$$

* Basset, p. 196.

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This vanishes for f_2 and f_3 and reduces to q for a scroll. For ϕ_4 it gives the correct value 10 (since each Kummer's cone is a quadric).

It is interesting to verify this result by a correspondence method, the dual of that given in § 20. Consider the scroll Σ of bitangents meeting a line l; the developable of contact is of class $\mathbf{m}_{2-2}$ (§ 5·2). If we associate pairs of planes whose contacts are on the same generator of Σ, the intersections of corresponding pairs generate a scroll of class ρ_{2-2}, by the self-duality of the scroll. The united planes are thus given by*

$$\mu_4 + 2\mathbf{q} = 2\mathbf{m}_{2-2} - 2\rho_{2-2}.$$

On reducing we obtain the above value for $\mathbf{q}$.

34·2. The order of the bitangential curve is

$$\rho = \int am' + a'm = am - 13a + 11n + 8\epsilon(a - n) \quad \ldots(89).$$

This agrees with f_n and reduces to ρ for scrolls. For ϕ_4 it gives the correct value 36, for the curve consists of 16 lines and the five curves of contact of the Kummer's cones†.

34·3. There is a number ϖ_1 of stationary tangent planes to the edge of regression of $\mathbf{B}$, whose cuspidal points are the intersections of the spinodal and bitangential curves. We have

$$\varpi_1 = \int m'\mathbf{c} + m\mathbf{c}' = 2m(2m - 2a + j) + 24(8a - 3j - 4m - 4n) \quad \ldots\ldots(90).$$

This agrees with f_n and vanishes for all the surfaces ϕ_n.

34·4. There is a number ϖ_3 of tritangent planes given by

$$\varpi_3 = \int \mathbf{b}m' + \mathbf{b}'m$$

$$= \binom{m}{3} + m(\tfrac{1}{2}\tfrac{1}{2}a - 3j - 6m) - 160a + 80j + 85m + 75n + \tfrac{1}{3}(2a + 2j + m) \quad \ldots\ldots(91).$$

This agrees with the formula for f_n and reduces to t for a scroll.

34·5. The order of the edge of regression is obtained (§ 4·6) from the equation

$$\Delta\mathbf{s} = 3mm'(m + m' - 3) - 6(m'\mathbf{b} + m\mathbf{b}') - 8(m'\mathbf{c} + m\mathbf{c}').$$

* The dual method is due to James (J. p. 216); the coefficient of n_4 should be unity in his result.

† Basset, p. 196.

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To obtain the particular cases, we note that $\mathbf{s}$ must vanish for f_2 (considered either as a scroll or a surface) and for a scroll it must reduce to s. Further, a pencil of planes has an edge of regression of order -3, since this is the value yielded by the general Cayley-Plücker formulae. Hence for f_3 we must have $\mathbf{s} = -81$ and for ϕ_4, $\mathbf{s} = -48$, since the Kummer's cones have no edge of regression.

Thus

$$\mathbf{s} = 5\binom{m}{2} + m(2j - a) + 4\tfrac{17}{2}a - 1\tfrac{77}{2}j - 107m - 102n - 48\epsilon(a - n) \quad \ldots\ldots(92).$$

This agrees with the general formula* for f_n.

34·6. The number of apparent double planes is given, as in § 4·5, by

$$\Delta\mathbf{h} = \tfrac{1}{2}mm'(m-1)(m'-1) + \mathbf{bb}' + m(m'-2)\mathbf{b}' + m'(m-2)\mathbf{b},$$

since, as is well known, the number $\frac{1}{2}mm'(m-1)(m'-1)$ represents the value of $\mathbf{h}$ for the developable common to two surfaces, whether they possess double and cuspidal developables or not. The result must agree with the formula for f_n and reduce to h in the case of a scroll.

We thus obtain

$$\mathbf{h} = 3\binom{m}{4} - 18\binom{m}{3} + \binom{m}{2}\{\tfrac{11}{2}a - 3j + 48\} + 4m(6j - 11a) + 1\tfrac{21}{8}a^2 - \tfrac{33}{2}aj + \tfrac{9}{2}j^2 + \tfrac{659}{4}a - 163j - 1\tfrac{13}{2}m - 80n \quad \ldots(93).$$

This satisfies all the required conditions.

34·7. The number of intersections of Γ with the bitangential curve is given by

$$\Delta i = \boldsymbol{\rho}n' + \boldsymbol{\rho}'n + a'mn + am'n' + \rho m' + \rho' m,$$

since the common tangent planes to F, F' touch F along a curve of order am' and F' along a curve of order $a'm$.

Hence

$$i = amn - 13an - am - 2jm - m^2 + 11n^2 + 2a + 25j + 13m - 11n \quad \ldots\ldots(94),$$

vanishing identically for f_n and giving 36 for ϕ_4.

The Flecnodal Developable.

35. Some formulae for $\mathbf{F}$ have already been obtained in § 12. We thus have for the order

$$\mu_4 = 11j - 10a + 6m - 4n + 4\epsilon n.$$

* Basset, p. 283; the coefficient -11 should be -12.

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The number ϖ_6 of hyperflecnodal planes is equal to μ_5, whence

$$\varpi_6 = 60j - 130a + 35m + 75n + 10\epsilon(3a - 2n).$$

The order of the curve of contact is

$$m_4 = 11a - 13n + \epsilon(6n - 5a).$$

35·1. To find the class of $\mathbf{F}$ we proceed by a method which is the dual of § 18·1. Consider the scroll $\mathbf{S}'$ of inflexional tangents to F at the points of contact of tangent planes passing through a given point P. As in § 18·1 we find that its class is $2m + \mu_3$; it meets F in the tangent cone from P and in a residual developable of class $m(2m + \mu_3) - 6m$, since the cone is double for $\mathbf{S}'$, and F and $\mathbf{S}'$ have inflexional intersection along it. The planes of this developable which pass through P consist of

(i) the required planes, ν_4 in number;

(ii) the $\mathbf{b}$ planes of $\mathbf{B}$, each counted β times, say;

(iii) the $\mathbf{c}$ planes of $\mathbf{C}$, each counted γ times;

(iv) the tangent planes to F containing the inflexional tangents through P, $\mu_3(m - 3)$ in number.

Thus $$m(2m + \mu_3) - 6m = \nu_4 + \beta\mathbf{b} + \gamma\mathbf{c} + \mu_3(m - 3).$$

Since ν_4 must be linear in the characters of F, we have $\beta = 4$. To find γ we consider f_3; the flecnodal curve consists of the 27 lines, and ν_4 is the number of intersections of this curve with the first polar of P; hence $\nu_4 = 54$. This gives $\gamma = 3$.

On simplifying we obtain

$$\nu_4 = 11m - 13a + 12j + \epsilon(6n - 5a) \quad \ldots\ldots\ldots(95).$$

The ϵ-terms are obtained from the fact that $\mathbf{F}$ and its curve of contact reciprocate into one another, so that for a scroll, ν_4 must reduce to m_4.

35·2. *The planes* ϖ_6. We may verify the known result by the correspondence method of § 33. Through any line l draw a plane π meeting the flecnodal curve in m_4 points P; the flecnodal tangent at P meets F in $n - 4$ further points P', giving planes π' which form with π a correspondence $((n-4)m_4, n\mu_4 - 4m_4)$, $n\mu_4 - 4m_4$ being the order of the residual intersection of $\mathbf{F}$ and F. The united planes occur for the planes ϖ_6, the $(n - 4)\mu_4$ lines PP' which meet l, and the points i_4 (§ 31·2). Hence

$$\varpi_6 + (n - 4)\mu_4 + i_4 = (n - 4)m_4 + n\mu_4 - 4m_4,$$

so that

$$\varpi_6 = 60j - 130a + 35m + 75n.$$

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36. *The biflecnodal planes* ϖ_4. We first determine the order μ_4' of the developable generated by the ordinary tangent at a flecnode of F. Consider the surface Φ mentioned in § 32; at a point where this meets the flecnodal curve, either the ordinary or the flecnodal tangent must meet the line l; hence

$$\mu_4 + \mu_4' = (3n - 4)\, m_4 - 3\,(i_4 + j).$$

On substituting we obtain

$$\mu_4' = 34j - 40a + 27m + 17n.$$

Now draw a plane π through l, meeting the flecnodal curve at m_4 points P; the ordinary tangent at P meets F in $n-3$ further points P', lying on the residual intersection of F and the developable μ_4', of order $n\mu_4' - 3m_4$. The planes π' through l and P' form with π a correspondence $((n-3)\, m_4,\ n\mu_4' - 3m_4)$ whose united planes are given by

(i) the planes ϖ_4, counted twice;

(ii) the planes ϖ_5;

(iii) the $(n-3)\,\mu_4'$ lines PP' meeting l;

(iv) the points i_4, counted once; and

(v) the j pinch-points, counted twice.

Thus

$$2\varpi_4 + \varpi_5 + (n-3)\,\mu_4' + i_4 + 2j = (n-3)\, m_4 + n\mu_4' - 3m_4,$$

which gives

$$\varpi_4 = 5\,(10j - 14a + 7m + 9n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (96).$$

37. *The planes* ϖ_2. The number of flecnodal planes which are elsewhere tangent is $\varpi_2 = \int \nu_4 m' + \nu_4' m$. Three constants of integration are found by comparison with the general formula* for f_n; the coefficient of j is obtained by substituting the characters of ϕ_n; if $J = -199$, the result reduces to $45n$, giving the correct value for the general cubic surface ϕ_3. Thus

$$\varpi_2 = (12j - 13a + 11m)\, n + 386a - 199j - 187m - 195n \quad \ldots (97).$$

38. It would be possible to obtain in like manner the dual of the correspondence equations given in §§ 22, 25. Since these results are of minor importance we merely give the formulae connected with the scrolls S_{2-2-2} and S_{3-2}.

* Basset, p. 273.

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Taking the simpler results first we have for the class of the developable of three-point contacts of S_{3-2} with F,

$$\mathbf{m}_{3:} = \int a'\mathbf{m}_3 + a\mathbf{m}_3'$$

$$= a(2j - a + 3m) + 48a - 40j - 48m + 2\epsilon(a - n) \quad \ldots\ldots(98).$$

The special cases are obtained from the fact that the result must reduce to $m_{3:}$ for a scroll so that, as before, $d^{(4)} = 4$. It must vanish for f_4 and give kl for f_3 and ϕ_4 ($k = -4$). As anticipated, the formula does not contain n.

38·1. The developable of ordinary contacts is of class

$$\mathbf{m}_{2:} = \int a'\nu_3 + m'\mu_3 \ldots$$

$$= 3a(a - n) + m(2j - a + m) + 48a - 36j - 48m + 6\epsilon(n - a)$$

$$\ldots\ldots(99).$$

As in §10·2 we have $d^{(4)} = 12$; we find that $k = -6$ and use the special cases already quoted.

38·2. The developable of contact of the scroll S_{2-2-2} is of class

$$\mathbf{m}_{2-2-2} = \int a'\nu_{2-2} + m'\mu_{2-2} + a'\mathbf{m}_{2-2} \ldots$$

$$= 3\binom{a}{3} + m\binom{a}{2} - \tfrac{3}{2}a^2 + a(4n - 6j - \tfrac{17}{2}m + 2\epsilon m)$$

$$- 3jm - 2m^2 - 113a + 100j + 128m - 4\epsilon n \ldots(100).$$

This vanishes for f_4 and f_5 and gives kl ($k = 8$) for f_3, ϕ_4, ϕ_5. It also gives the correct value $120 = 16k + d^{(4)}$ for F_4, since $d^{(4)} = -8$, as in §6·1.

39. We give here the line constants which have been used in the previous work; $d^{(3)}$ is found by applying the scroll formula to the cubic scroll (3, 4) which contains a nodal and an ordinary line (the directrix): the formulae then give the appropriate values of $k + d^{(3)}$. The conjectured values of $d^{(5)}$ are shown in italics; those for $m_{2:}$ and $m_{3:}$ satisfy a known relation (§ 10·2) and similarly for $\mathbf{m}_{2:}$ and $\mathbf{m}_{3:}$. Apart from these, it is sufficiently clear that k and $d^{(n)}$ are connected by a simple formula; but at present it is impossible to say what that formula may be.

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	k	$d^{(3)}$	$d^{(4)}$	$d^{(5)}$
δ_{2-2}	—	4	−4	4
δ_3	—	−3	6	−9
m_{2-2-2}	4	8	−8	8
$\mu_{2-2-2-2}$	16	16	−16	16
μ_{3-3}	9	9	−18	36
μ_{3-2-2}	−36	−36	48	−60
$m_{3:}$	−2	−4	4	−4
$m_{2:}$	−3	−6	12	−18
m_4	1	2	10	—
μ_{4-2}	16	16	−32	48
μ_5	−5	−5	20	—
$\mathbf{m}_{2-2-2}$	8	8	−8	8
$\mathbf{m}_{3:}$	−4	−4	4	−4
$\mathbf{m}_{2:}$	−6	−6	12	−18

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