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Quasistatic evolution of a dissipative plasma column in vacuum: errata and addenda

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Abstract. We provide a revised, and substantially enriched, version of an old study of ours, dealing with the theory of the slow evolution of a strictly dissipative, cylindrical plasma column in vacuum [*J. Plasma Phys.* **43**, 217–237 (1990)]. After correcting a technical mistake made in our previous formulation of the model of the physical system under consideration, we develop a fully amended version of the associated theory, paying special attention to the problem's well-posedness. A number of comments, generalizations and further details have also been added.

1. Introduction

A long time ago, the present author faced the problem of calculating the evolution (assumed sufficiently slow, and 'quasistatic') of a cylindrical (z-independent) physical system consisting of (i) a strictly dissipative plasma region with simply connected normal section (or a 'column'), and (ii) a surrounding vacuum region with annular normal section, bounded in turn by a fixed wall where the tangential electric field had to be (arbitrarily) given. (Lo Surdo 1990, hereinafter referred to as LS90).

Needless to say, our study was intended to better understand the 'behaviour' (in both space and time) of a large-aspect-ratio tokamak discharge close to a 'self-consistent' (the precise meaning of this attribute will be explained later), sufficiently stable, but not necessarily static, equilibrium.

A convenient model of the system was elaborated and analysed in LS90 with this purpose, but unfortunately a couple of technical mistakes (not quite independent of each other) were made. Namely, first, we used too strong an accessory condition at the plasma–vacuum interface, and, secondly, we did not recognize the singular character of an evolutionary 'parabolic-like' (the precise meaning of this attribute too will be explained later) $(1+1)$-dimensional system at one extreme of its 'space' domain (a real interval).

Even though most of LS90 remains valid regarding the model and its analysis, the above errors had very serious consequences for our final results. Actually, the full correction of the theory was not exceedingly difficult (after all), but, unfortunately again, our study had already been published when, a little later, its wrong assumptions/conclusions were discovered and repaired. Now we find it opportune (not to say dutiful, the unpardonable delay apart) to devote a new paper to (i) scrutinizing the errors of LS90 and the unlucky chain of their logical consequences, (ii) providing the full, correct version of the

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theory, and (iii) enriching it with a number of further considerations, comments and details.

In doing so, and for the reader's convenience, all equations and formulae from LS90 keep their numbering, preceded by 'I', and all symbols retain their original meaning. Furthermore, all equations and formulae from LS90 that are presented here in their corrected versions are indicated with an asterisk*. It goes without saying that the understanding of the present study presupposes full acquaintance with (and continuous recourse to) LS90. For convenience, a complete list of symbols will be found at the end of the present text.

As was to be expected, the most exotic mathematical aspects of the evolution problem under consideration disappear on assuming circular – besides cylindrical – symmetry, mainly because the 'magnetic average' operator reduces to the identity and all 'space' differential operators become ordinary. The analysis of such a highly symmetric (however artificial) configuration turns out to be very instructive, and can be entirely carried out by relatively simple means. This will be illustrated in a separate, forthcoming paper.

After LS90 was published, the stationary ($\partial_t \equiv 0$) state of the same physical system was investigated (Lo Surdo 1994) – of course with the unsuitable interface condition used in LS90 having been corrected – and, after that, a number of special applications were made (Lo Surdo et al. 1996; Guo et al. 1997; Lo Surdo and Guo 1998). For brevity, the above (generally non-static) equilibria have been conventionally termed 'self-consistent', because they account for the full set of (stationary) plasma MHD equations, in the same approximation we have used both in LS90 and here. In particular, the stationary circular-cylindrical equilibrium has been fully analysed in Lo Surdo and Guo (1998, Sec. 4).

2. The two basic errors in Lo Surdo (1990)

Almost everything wrong in LS90 was a logical consequence of the two technical errors described below.

(a) The local, kinematic interface (plasma–vacuum) condition

$$v_n^- - u = 0 \tag{I2.19}$$

(see the list of symbols at the end of this paper) was assumed. Although correct at first sight, it turns out to be too stringent, and has to be conveniently weakened (see below).

(b) We did not notice that the quasilinear evolution system (I6.6a), (I6.2′) for (f, T) in the independent domain $\{\psi \in (\psi_{\mathbf{a}}(t) < 0, 0)\} \times \{t \geqslant 0\}$ was singular at the left extreme†

2.1. Consequences of (a)

From (I2.19), (I4.29) followed, and this (reasonably) implied (circular symmetry apart) the three separate conditions

$$f_{\psi}^- = 0, \tag{I4.27}$$

$$p_{\psi}^- = 0, \tag{I4.28}$$

$$T_{\psi}^- = 0. \tag{I4.30}$$

† Actually, (I6.6a) and (I6.2′) will be replaced by their amended versions (I6.6a*) and (I6.2′*), both of which, however, share the same type of singularity.

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Then we got $H^- = 0$, and so $U = 0$.

Furthermore, as a consequence of conditions (I4.27), (I4.28) and (I4.30), not only the (local) *scalar* conditions $q_n^- = 0$ and $J_n^- = 0$ (see LS90, footnote ‡ on p. 220), but also the stronger (local) *vector* conditions

$$\mathbf{q}^- - \mathbf{q}^-_{T_\parallel} = 0, \tag{I2.17}$$

$$\mathbf{J}^- = 0, \tag{I2.18}$$

can be proved† to be automatically fulfilled (also taking into account $U = 0$). The conditions (I2.17) and (I2.18) apart, we stress that, whereas $J_n^- = 0$ is a trivial identity, $q_n^- = 0$ again turns out to be too stringent, and has to be weakened in a similar way as (I2.19); see below.

2.2. Consequences of (*b*)

Having missed the singular nature of the system (I6.6a), (I6.2′) in LS90, we were forced to complement it with four independent side-conditions (SCs). Since these SCs had to be prescribed on γ (obviously), and three of them were already available (i.e. (I4.27), (I4.28) and (I4.30)), we added condition (I2.20).‡ In the light of the ascertained singularity of (I6.6a)/(I6.6a*) and (I6.2′)/(I6.2′*), a serious rethinking of the well-posedness problem is in order (see Sec. 3).

The following remark is worth making before closing this section. By definition, $U \equiv \partial_z \Phi$, (I4.6), is a function of t at most. Similarly, $\psi|_\gamma$ turns out a function of t at most (LS90, p. 227, line 8), say $V = V(t)$. Under the gauge relations (I4.8a), one is always allowed to require that $U' = \text{const}$ (and in particular $U' = 0$), by choosing $\psi' = \psi - \int dt\, U$.§ Alternatively, one can require $\psi'|_\gamma = \text{const}$ (and in particular $\psi'|_\gamma = 0$), by choosing $\psi' = \psi - V$.

As we do here, we took the second option in LS90, thus defining γ through $\psi'|_\gamma$ (or rather $\psi|_\gamma) = 0$, and letting U' (or rather U) to be an a priori generic function of t. After conditions (I4.27), (I4.28) and (I4.30) were erroneously taken to be necessary, and so a posteriori, U was forced to vanish. But with the above conditions dropping out (as we shall see in a moment), U will no longer be required to vanish.

3. The correct equations

As long as the determination of (ψ, f, p, T) is concerned, the mass and energy (local) balances, (I2.5) and (I2.7) respectively, enter our analysis only through their magnetic averages, (I6.2) and (I6.3) respectively.‖ The correct interface conditions to complement these magnetically averaged equations must be induced starting from them, and not from the original local balances. This is tantamount to replacing the local conditions $v_n^- - u = 0$ and $q_n^- = 0$, equivalent

† This is not quite immediate, but straightforward.

‡ There is a trivial oversight (without consequences) in condition (I2.20), because the given δ cannot depend on γ.

§ Again due to an oversight (without consequences), this relation was written as $\psi' = \psi + Ut$ in LS90 (p. 228, line 2).

‖ In these equations, U was kept equal to zero, so they will be corrected in this sense; see (I6.2*) and (I6.3*).

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to $(\partial_t \psi + \mathbf{v}\cdot\boldsymbol{\nabla}\psi)^- = 0$, and to $(\mathbf{q}\cdot\boldsymbol{\nabla}\psi)^- = 0$ respectively, by their magnetic averages

$$\langle \partial_t \psi + \mathbf{v}\cdot\boldsymbol{\nabla}\psi \rangle^- = 0, \tag{3.1}$$

$$\langle \mathbf{q}\cdot\boldsymbol{\nabla}\psi \rangle^- = 0. \tag{3.2}$$

Plainly, the physical meaning of these equations is that neither mass ($\rho^- > 0$!) nor heat escapes from the plasma as a whole.†

The replacement of (I2.19) by (3.1) has immediate and important consequences. Namely, now $\langle H - U\rangle^-$, and not H^- and U separately as before, must vanish: then conditions (I4.27), (I4.28) and (I4.30) drop out all together, and, with them, $U = 0$ drops out too. The last fact, in particular, requires us to restore U in a number of equations of LS90 that had been written after $U = 0$ was erroneously recognized as a necessity. To begin with, we have (I6.1)–(I6.3), (I6.6a), (I6.7a) and (I6.2′), as well as the definition of Q (see LS90, p. 228, lines 17 and 18). Such a restoration of U is straightforward (but cumbersome), and we get

$$\langle \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}s\rangle - f\langle w - U\rangle = \langle K^{\parallel}\rangle \equiv \delta^{(c)} \tag{I6.1*}$$

in place of (I6.1),

$$(T_t' + UT_\psi')\partial_t\rho + (p_t + Up_\psi)\partial_p\rho - \rho\mathscr{A}\circ(w-U) = \mathscr{A}\circ(\rho H) \equiv \delta^{(m)} \tag{I6.2*}$$

in place of (I6.2),

$$p_t + Up_\psi - \tfrac{5}{3}p\mathscr{A}\circ(w-U) - \tfrac{2}{3}\langle S\rangle = \tfrac{5}{3}\mathscr{A}\circ(pH) + \tfrac{2}{3}f_\psi\langle K^{\parallel}\rangle - \tfrac{2}{3}\mathscr{A}\circ L \equiv \delta^{(e)} \tag{I6.3*}$$

in place of (I6.3),‡

$$f_t + Uf_\psi = f\mathscr{A}\circ(w-U) + \mathscr{A}\circ K^{\parallel} \tag{I6.6a*}$$

in place of (I6.6a),

$$(\nabla^2 - j_\psi)(w-U) + [h\mathscr{A}\circ(w-U)]_\psi = -\tfrac{2}{3}\langle S\rangle_\psi - (\delta^{(e)} + f\mathscr{A}\circ K^{\parallel})_\psi \equiv \Sigma \tag{I6.7a*}$$

in place of (I.6.7a),

$$T_t' + UT_\psi' = (\partial_T\rho)^{-1}[k\mathscr{A}\circ(w-U) - \partial_p\rho(\tfrac{2}{3}\langle S\rangle + \delta^{(e)}) + \delta^{(m)}] \tag{I.6.2′*}$$

in place of (I6.2′), and

$$Q \doteq f(w-U) - \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}s + K^{\parallel} \tag{3.3}$$

in place of the corresponding definition in LS90.

In contrast, (I4.6), (I4.7), (I4.13)–(I4.15) and (I4.31), all of which contain U, are correct, since they had been written before the erroneous request $U = 0$ was made. Also, (I4.17)–(I4.25) remain valid; (I4.19a) and (I4.25), in particular, can be put into the equivalent forms

$$[\![w - U]\!] = 0, \tag{I4.19a*}$$

$$[\![\partial_n(w-U)]\!] - \mu^-(w^0 - U) = \mu^- U, \tag{I4.25*}$$

† In particular, because of (I2.5), the first condition means that the total plasma mass does not vary with time (use the general relations (I5.5), noting that the $\langle\partial_t W\rangle$ under the integral sign was misprinted as $\langle\partial_t\psi\rangle$).

‡ The + sign in front of $\frac{2}{3}\mathscr{A}\circ L$ on the right-hand side of (I6.3) is a misprint.

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where we have written $-j/|\nabla\psi|$ as μ for brevity. However, the simplified version (I4.25′) of (I4.25) can no longer be used, since it is a consequence of (I4.27) and (I4.28).

One more equation must similarly be modified: on restoring the (no longer vanishing) U and $\langle K^{\parallel}\rangle$ in (I6.10), we get†

$$\mathscr{P}|\Omega_V|^{-1} = [\mathscr{J}^0|\Omega_V|^{-1} + (\mathscr{J}^0)^{-1}\mathscr{J}^-_\psi][f^0\langle w^0 - U\rangle + \langle K^{\parallel}\rangle^{-1}]$$
$$+ f^0\langle w - U\rangle^-_\psi + \langle K^{\parallel}\rangle^-_\psi - Uf^-_\psi. \qquad \text{(I6.10*)}$$

This concludes the list of the modifications that arise from having replaced (I2.19) by (3.1). With regard to (3.2), this will be used as such in what follows.

4. Other comments, generalizations, etc.

(a) For brevity, in LS90, we did not give explicitly the continuity classes to which the various unknowns should a priori belong, in particular as functions of $\boldsymbol{\xi} \in \Omega \equiv \Omega_P \cup \Omega_V$ ($\equiv$ at a fixed t). For instance, it is obvious that both ψ and s should be $C^2(\Omega)$ and that φ should be $C^2(\Omega_V)$. Furthermore, all the relevant functions must be required to have unique continuous extensions to $\partial\Omega \equiv \gamma \cup \Gamma$ (not necessarily the *same* extension to the two sides of γ). As far as the dependence on t ($\equiv$ at a fixed $\boldsymbol{\xi}$) is concerned, it is sufficient to require the relevant unknowns to be $C^1(t > 0)$.

(b) Equation (I4.13) contains two misprints: a wrong sign in front of r on the left-hand side, and a $\boldsymbol{\nu}$ in place of $\boldsymbol{\kappa}$ in the second set of square brackets on the right-hand side. It should be noted, however, that the above term in r does not play any significant role from the standpoint of the present model analysis. Ignoring this term is nothing other than the so-called 'drift approximation'.

(c) If the annular region between the plasma and the fixed wall Γ is assumed to be filled by some gas (instead of vacuum),‡ the (uniform) pressure in it must also be given, say as $\bar{p} = \bar{p}(t) > 0$. Then (I2.16) will be replaced by

$$p^- = \bar{p}(t), \qquad \text{(I2.16*)}$$

and (I6.8) will be modified accordingly (simply by adding $\bar{p}(t)$ to its right-hand side), to (I6.8*), say. However, in what follows we shall keep on using (I6.8) for the sake of simplicity.

(d) Condition (3.2) has been chosen because of its conceptual simplicity, but, of course, it is not very realistic. However, this condition could easily be modified, for instance by prescribing $\oint d\gamma\, q_n^-$ as a function of t, or as a quantity proportional to the difference $T^- - T^+$, with T^+ given as a function of t at most, and so on. Again, for simplicity, here we shall keep on using condition (3.2).

† With no risk of being misunderstood, we have written $\langle\ \rangle^-_{\bar\psi}$ for $(\langle\ \rangle_\psi)^-$, and the like.

‡ Possibly, in this case we should denote it by Ω_g. We recall that in our model the curve γ roughly represents some rapid transition region between plasma and gas (or vacuum).

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(e) Equation (I6.5b) (in Ω_V), and (I6.4a) (in Ω) can be put into the (trivially) equivalent forms

$$\nabla^2(w-U)=0, \tag{I6.5b*}$$

$$\nabla^2 s = f_t + Uf_\psi + f_\psi(w-U). \tag{I6.4a*}$$

With these last formal modifications, we see that w only appears through the difference $w_* \doteq w-U$ wherever it is found in our equations (including (I4.31)), and can be eliminated in favour of w_* everywhere. U still independently enters (3.1), (I6.10*) and (I4.25*) and the set of equations (I6.2*), (I6.6a*), (I6.4a*), (I6.2′*) – in the second case through sums of type $X_t + UX_\psi$, where X is one of the functions f, p or T.† With our choice $\psi|_\gamma = 0$, the physical meaning of $-U$ is that of the z component of the electric force acting on a unit positive charge 'in normal motion' with γ (i.e. moving with the normal speed $u = -(\partial_t\psi/|\nabla\psi|)^0$).

(f) In LS90, S was thought of as a given function of $(\boldsymbol{\xi}, t)$ and of $(p, T, |\mathbf{B}|^2)$; see p. 222, below (2.7). Often such a degree of generality is redundant from a physical standpoint: we could limit ourselves to treat S as a given, sufficiently regular function of the plasma state (p, T). In this case, $\langle S\rangle$ can be replaced by S everywhere.

(g) The proof of the single-valuedness of φ in Ω given in LS90 could, and should be improved. First of all, as we have seen in (a), φ has to be $C^2(\Omega_V)$, so $\oint_l d\mathbf{l}\cdot\nabla\varphi$ has the same value for any cycle l of Ω_V reducible to Γ; hence, in the limit, we get

$$\oint d\boldsymbol{\Gamma}\cdot\nabla\varphi|_\Gamma = \oint d\boldsymbol{\gamma}\cdot\nabla\varphi|^+.$$

However, we are not allowed to similarly require that $\varphi\in C^2(\Omega_P)$: what is legitimate is simply to require a priori that $\varphi\in C^1(\Omega_P\backslash\{\mathbf{a}\})$ (a non simply connected domain!), and that $\nabla\varphi$ be single-valued there. This fits with the requirements that both ψ and s are $C^2(\Omega_P)$ (see (a)), also taking into account the first-order ordinary differential equation φ must satisfy. Now it is not difficult, although a little cumbersome, to prove that, as $\boldsymbol{\xi}\to\mathbf{a}$ in any way, $Q = O(|\nabla\psi|)$, and so

$$\int_0 dl\,\frac{Q}{|\nabla\psi|}(\psi, l)\to 0$$

like $|\nabla\psi|$ for $\boldsymbol{\xi}\to\mathbf{a}$ in any way, or rather for $\psi\to\psi_\mathbf{a}+$, and hence that $\varphi(\psi_\mathbf{a}+, l) = k^{(\varphi)}(\psi_\mathbf{a}+)$. Of course, we shall choose such a $k^{(\varphi)}$ that the second limit exists and is finite; then we conclude that φ has a unique continuous extension to the whole of Ω_P. Now it can be proved that a function (like φ) endowed with the three above properties (i.e. that belongs to $C^1(\Omega_P\backslash\{\mathbf{a}\})$, has single-valued gradient in the same $\Omega_P\backslash\{\mathbf{a}\}$, and has a unique continuous extension to Ω_P) must be single-valued in Ω_P, or $\oint_l d\mathbf{l}\cdot\nabla\varphi = 0$ for every cycle l of $\Omega_P-\{\mathbf{a}\}$ reducible to γ, and, in the limit, for $l=\gamma$. [This theorem becomes trivial when Ω_P reduces to an

† The last circumstance could suggest that we introduce a length coordinate along the lines $d_t\psi = U$ in the (t,ψ) plane as independent variable in place of t; however, it can be seen that no formal nor substantial advantage would follow from this.

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open disk (say, with unit radius), and **a** is its centre. Indeed, since $\boldsymbol{\nabla}\varphi$ is single-valued by assumption, the most general form of φ must be (with (r,θ) being the usual polar coordinates)

$$\varphi(r,\theta)=\varphi_0(r,\theta)+\varphi_1(r)\theta \quad \text{for } 0<r<1,\ 0\leqslant\theta<2\pi,$$

where φ_0 is single-valued and $\varphi(0+,\theta)$ (and hence both $\varphi_0(0+,\theta)$ and $\varphi_1(0+)\theta$) must exist and be finite and independent of θ, so $\varphi_1(0+)=0$. Taking the r-derivative of φ, and requiring its single-valuedness, we get $d_r\varphi_1=0$, or $\varphi_1=\text{const}$, for $0<r<1$. The constant must be zero because of the above requirement, and the theorem is thus proved in this special case. Appendix B of LS90 can now be ignored.]

Owing to the continuity of φ across γ, (I4.21′), we conclude that φ is everywhere single-valued in $\Omega\backslash\{\mathbf{a}\}$, or

$$\oint dl\,\frac{Q}{|\boldsymbol{\nabla}\psi|}=0$$

for every cycle l of $\Omega\backslash\{\mathbf{a}\}$ reducible to Γ, and, in the limit, for $l=\Gamma$.† The condition $\oint d\Gamma\,\partial_\Gamma\varphi=0$ then allows us to move to a (unique) new gauge (φ',s') for which $\partial_\Gamma\varphi'=0$, or rather $\varphi'|_\Gamma=0$, according to what has been explained in LS90, p. 228, lines 6–13.

5. Well-posedness of the evolution problem

We now come to the very core of the discussion, the (possibly conditioned) well-posedness of our evolution problem. For the sake of previty, let us rewrite the expression (I4.16) for H in the form $\eta_\parallel ff_\psi+\mathscr{N}p_\psi+\mathscr{L}T_\psi$, and that for L (LS90, top of p. 229), in the form $\mathscr{Z}p_\psi+\mathscr{M}T_\psi$. This will unequivocally define the expressions for $\mathscr{N},\mathscr{L},\mathscr{Z}$ and $\mathscr{M}$. Stating, following Onsager (1931), that $\boldsymbol{\nu}$ and $\boldsymbol{\kappa}$ are proportional means that a local temperature $T^*>0$ exists such that $\boldsymbol{\nu}=T^*\boldsymbol{\kappa}$. In a classical plasma, this T^* roughly equals the electron temperature (see e.g. Balescu 1988, Chap. 5, pp. 230ff.). In our one-fluid plasma, we shall obviously have $T^*=T$, so $\boldsymbol{\nu}=T\boldsymbol{\kappa}$ and $\mathscr{Z}=-T\mathscr{L}$.

At this point, it is worth giving a better specification of the precise requirements behind the attribute 'dissipative' (or rather 'strictly dissipative') appearing in the title of this paper. The 'dissipation function' $\mathscr{D}$ in our plasma, where we have no other generalized forces other than $\mathbf{E}'$ ('effective electric field') and $\boldsymbol{\nabla}T$, is

$$\mathscr{D}\doteq\frac{1}{T}\left(\mathbf{E}'\cdot\mathbf{J}+T\mathbf{q}\cdot\boldsymbol{\nabla}\frac{1}{T}\right)$$

(see e.g. Hughes and Young 1988, Chap. 5, p. 132). On thermodynamic grounds, $\mathscr{D}\geqslant 0$, since it equals the specific entropy production rate per unit time. With linear constitutive relations (such as those we have chosen; see (I2.1) and (I2.2)) $\mathscr{D}$ turns out to be a bilinear form in the generalized forces (or generalized currents, under a non-degeneracy condition); so this form is universally positive-semidefinite. Strict dissipativity simply means that $\mathscr{D}$ is positive-definite.

† $\boldsymbol{\nabla}\varphi$ can also be proved to vanish like $|\boldsymbol{\nabla}\psi|$ for $\boldsymbol{\xi}\to\mathbf{a}$ in any way.

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It is known, and can be proved by some cumbersome algebra, that in our case strict dissipativity is equivalent to

$$\sigma_{\parallel} > 0, \tag{5.1a}$$

$$\sigma_{\perp} > 0, \tag{5.1b}$$

$$\lambda_{\parallel}\sigma_{\parallel} - \kappa_{\parallel}\nu_{\parallel} > 0, \tag{5.2a}$$

$$\lambda_{\perp}\sigma_{\perp} - \kappa_{\perp}\nu_{\perp} > 0. \tag{5.2b}$$

The inequalities (5.1a,b) are sufficient to ensure the existence of the resistivity tensor $\boldsymbol{\eta}$ (unique) inverse of $\boldsymbol{\sigma}, \boldsymbol{\eta} \doteq \boldsymbol{\sigma}^{-1}$.† From Onsager's relations, the inequalities (5.2a,b) require $\lambda_{\parallel} > 0, \lambda_{\perp} > 0$. In our more specific case, where we treat $\lambda_{\parallel}$ as a formal (positive) infinity, we can forget the inequality (5.2a).

It is worth examining the asymptotic model $\lambda_{\parallel} \to +\infty$ a little deeper. As is well known, according to classical transport theory, $\lambda_{\wedge}/\lambda_{\parallel} = O(x^{-1})$ and $\lambda_{\perp}/\lambda_{\parallel} = O(x^{-2})$, where x is a measure of the (very large) magnitude of the magnetic field. Hence, by requiring that both $\lambda_{\wedge}$ and $\lambda_{\perp}$ remain finite as $x \to +\infty$, $\lambda_{\parallel}$ must diverge with x (at least like x^2); but, in this case, if we want to prevent $-\mathbf{q}_{T_{\parallel}} \doteq \lambda_{\parallel}\hat{\mathbf{B}}\hat{\mathbf{B}}\cdot\boldsymbol{\nabla}T$ from diverging in turn, $\hat{\mathbf{B}}\cdot\boldsymbol{\nabla}T$ must vanish like $\lambda_{\parallel}^{-1}$. In this approximation, we have introduced the additional scalar unknown $\mathring{q}_{T_{\parallel}} \doteq \hat{\mathbf{B}}\cdot\mathbf{q}_{T_{\parallel}}$, then 'balancing' it by the additional scalar equations $\hat{\mathbf{B}}\cdot\boldsymbol{\nabla}T = 0$. This argument, to be found in LS90, is not quite free of criticism. What we actually do, indeed, consists in limiting ourselves to consider situations where the cosine $\hat{\mathbf{B}}\cdot\boldsymbol{\nabla}T/|\boldsymbol{\nabla}T|$ is so small as to make $|\mathbf{J}_{T_{\parallel}}| \ll |\mathbf{J}_{\parallel}|$ ($\mathbf{J}_T$ is the part of $\mathbf{J}$ driven by $\boldsymbol{\nabla}T$), but not necessarily such that $|\mathbf{q}_{T_{\parallel}}| \ll |\mathbf{q}_{\parallel}|$. It is plain that such a condition cannot be arrived at by merely exploiting properties (or asymptotic properties) of conductivities $(\boldsymbol{\sigma}, \boldsymbol{\kappa}, \boldsymbol{\nu} = T\boldsymbol{\kappa}, \boldsymbol{\lambda})$; it involves the generalized local forces $\mathbf{E}'$ and $\boldsymbol{\nabla}T$ as well, and hence the local plasma configuration.

Let us now return to strict dissipativity, or positive-definiteness of $\mathscr{D}$. In our model, taking for granted that $\sigma_{\parallel} > 0$, $\sigma_{\perp} > 0$, $\mathscr{D}$ can be expressed in terms of three generalized forces or currents only: for instance, as (a bilinear form) of the independent (arbitrary) generalized currents‡

$$j \equiv J_z, \qquad \mathbf{J}\cdot\hat{\mathbf{z}}\times\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|} = -f_{\psi}|\boldsymbol{\nabla}\psi| \equiv J_l, \qquad \mathbf{q}\cdot\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|} \equiv q_n.$$

For convenience, here we shall use f_{ψ} as generalized current in place of J_l.

† The necessary and sufficient conditions for the existence of $\boldsymbol{\eta}$ are $\sigma_{\parallel} \neq 0$ and $\bar{\sigma}^2 \doteq \sigma_{\wedge}^2 + \sigma_{\perp}^2 > 0$. In LS90, we required $\sigma_{\parallel}, \sigma_{\wedge}$ and $\sigma_{\perp}$ to all be positive, which is a little redundant to ensure the existence of $\boldsymbol{\eta}$.

‡ One could object that j is not quite arbitrary, since it must equal the Laplacian of ψ, which in turn is not a completely arbitrary function: as we know, ψ must be $C^2(\Omega_P)$, it must have a positive (or negative) Laplacian at $\mathbf{a}$, and its gradient must be different from zero in $\Omega_P \backslash \{\mathbf{a}\}$ and must vanish at $\mathbf{a}$ (we shall refer to such a ψ as a 'well-shaped' poloidal flux). However these requirements deal with the dependence of j on $\boldsymbol{\xi}$ in Ω_P, and not with the value of j at a given $\boldsymbol{\xi} \in \Omega_P$. Moreover, j and J_l do not seem to be fully independent, as is shown by the fact that reversing the j field is equivalent to reversing the J_l field. But the latter kind of dependence is trivial, being due to the convention of orienting $\hat{\mathbf{l}}$ like $\hat{\mathbf{z}}\times\boldsymbol{\nabla}\psi$; and, in any case, the reversal of the j field can be excluded by convention, as we did in LS90, p. 220, lines 10–11. The parities with respect to $\mathbf{B}_P$ of a number of fields of interest are as follows: ψ, j, U, w_*, $K^{\parallel}$, $K_z^{\perp}$, E_z, E_z' and $v_{\perp z}$ are odd, and f, p, T, s, φ, $\mathbf{K}_M^{\perp}$, $\mathbf{E}_M$, $\mathbf{E}_M'$ and $\mathbf{v}_{\perp M}$ are even.

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Some algebra then proves that the (symmetric) 3×3 matrix of $\mathscr{D}$ (as a bilinear form in (j, f_ψ, q_n)) is

	j	f_ψ	q_n
j	$T\mathscr{N}+\mathscr{M}^{-1}\mathscr{Z}^2$	$-T\mathscr{H}+\mathscr{M}^{-1}f\mathscr{Z}^2$	0
f_ψ		$T\mathscr{N}^*h^2+\mathscr{M}^{-1}f^2\mathscr{Z}^2$	0
q_n			$-\mathscr{M}^{-1}h^2$

For brevity, here we have written $|\boldsymbol{\nabla}\psi|$ as h and

$$\mathscr{N}^* \doteq (\eta_\parallel h^2+\eta_\perp f^2)|\mathbf{B}|^{-2}, \quad \mathscr{H} \doteq (\eta_\parallel-\eta_\perp) f h^2 |\mathbf{B}|^{-2}.$$

The (necessary and sufficient) conditions to have a positive $\mathscr{D}$, if $\boldsymbol{\xi}\neq\mathbf{a}$, are

$$-\mathscr{M}h^2>0, \tag{5.3a}$$

$$-T\mathscr{M}\mathscr{N}^*h^2-\mathscr{Z}^2f^2>0, \tag{5.3b}$$

$$\eta_\parallel(-T\mathscr{M}\eta_\perp h^2-\mathscr{Z}^2|\mathbf{B}|^2)>0. \tag{5.3c}$$

Both $\eta_\parallel$ and $\eta_\perp$ are greater than zero, so (5.3c) implies (5.3a,b), which can be ignored. A little algebra then shows that (5.3c) is equivalent to

$$\lambda_\perp\eta_\perp-\boldsymbol{\eta}\otimes\boldsymbol{\kappa}(\eta_\perp\nu_\wedge+\boldsymbol{\eta}\otimes\boldsymbol{\nu})-\boldsymbol{\eta}\odot\boldsymbol{\kappa}\eta_\perp\nu_\perp>0,$$

or else to $\lambda_\perp\sigma_\perp-\kappa_\perp\nu_\perp>0$, i.e. again to the inequality (5.2b). In conclusion, we do not find anything more than the result issuing from the general inequalities (5.2a,b) under the asymptotic condition $\lambda_\parallel\to+\infty$. Finally, at the magnetic axis, $\mathscr{D}$ reduces to $\eta_\parallel j^2/T$, and so $\mathscr{D}>0$ is automatically ensured because $j\neq 0$ (specifically, $j>0$ according to our convention).

Let us now assume that the unknown function ψ of t and of the 'space' variable $\boldsymbol{\xi}$, and the unknown functions f, p and T of t and ψ, as well as the unknown function U of t, be given at the 'present' time (say, $t=0$) under the full set of synchronous constraints they must satisfy, namely, under those constraints that involve only their values at $t=0$ (so, in particular, that do not contain their t derivatives).† A typical, and most important, synchronous constraint is the Grad–Shafranov equation (I4.11), linking ψ, f and p to each other.

In what follows in this section, we shall neither mention nor write down the present time $t=0$ explicitly. By use of (I4.11), or rather of (I6.8), $p=p(\psi)$ will be (or will have to be thought of as being) eliminated in terms of $j=j(\psi)$ (in $(\psi,0)$), $f(\psi)$ and $f^0\doteq f(\psi=0-)$. On doing so, in particular, in the interface conditions (3.1) and (3.2), one gets

$$f^0\langle\eta_\parallel-\mathscr{N}\rangle^-f_\psi^-+\langle\mathscr{L}\rangle^-T_\psi^-=\langle\mathscr{N}\rangle^-j^-+U, \tag{3.1$'$}$$

$$-f^0\langle\mathscr{Z}\rangle^-f_\psi^-+\langle\mathscr{M}\rangle^-T_\psi^-=\langle\mathscr{Z}\rangle^-j^-. \tag{3.2$'$}$$

Equation (5.3a$'$) can be used to eliminate U, and will be forgotten from now on together with U (as an unknown).

With the present $\psi=\psi(\boldsymbol{\xi}\in\Omega_P)$, $f=f(\psi\in(\psi_\mathbf{a},0))$, $T=T(\psi\in(\psi_\mathbf{a},0))$ given under the remaining synchronous constraints (e.g. (3.2$'$)), we shall now consider the linear problem for w_*, in Ω, consisting of

† Among these synchronous constraints, some have a qualitative character, like 'ψ is a well-shaped poloidal flux'.

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(i) (I6.7a*) in Ω_P;†

(ii) (I6.5b*) in Ω_V;

(iii) (I4.19a*) across γ;

(iv) (I4.25*) across γ;

(v) (I4.31) along Γ, or else (I4.31*) $w_*|_\Gamma = \alpha$.

We stress that p and U must be thought of as having been eliminated from the above equations, through (I6.8) and (3.1′) respectively.

The 'forcing' terms in this linear problem are $\Sigma = \Sigma(\psi)$ (see (I6.7a*), with $S = S(p, T)$ being given by assumption), $\mu^- U$ (see (I4.25*)) and α (see (I4.31*)). Note that Σ and $\mu^- U$ are known from the (constrained) present data, and α is given by assumption. Apart from being linear, the problem for w_* is fairly exotic in nature, partial-differential but also singular-integral, owing to the term $(h\mathscr{A} \circ w_*)_\psi$ on the left-hand side of (I6.7a*). Indeed, this term involves the first and second ψ derivatives of the integral, with measure $dl/|\nabla\psi|$ and over a singular subset (the closed curve $\psi_{\mathsf{a}} < \psi = \text{const} < 0$ in the ξ plane) of Ω_P, of the unknown w_*.

Now we shall assume that the above linear problem for w_* is non-degenerate (has no proper solutions), i.e. that‡

$$(\Sigma \equiv 0, \mu^- U \equiv 0, \alpha \equiv 0) \Rightarrow w_* = 0. \tag{5.4}$$

The obvious meaning of this implication is that, if a solution $w_* = w_*(\xi)$ exists (as we shall assume) then it must be unique, and expressible as the sum of a linear transform of Σ over $(\psi_{\mathsf{a}}, 0)$, say $\int_{\psi_{\mathsf{a}}}^0 d\psi\, \mathsf{A}(\xi, \psi)\Sigma(\psi)$, a linear transform of $\mu^- U$ over γ, say $U \oint_\gamma d\gamma\, \mathsf{B}(\xi, \gamma)\mu^-(\gamma)$, and a linear transform of α over Γ, say $\oint d\Gamma\, \mathsf{C}(\xi, \gamma)\alpha(\gamma)$. For the given Γ, the three kernels $(\mathsf{A}, \mathsf{B}, \mathsf{C})$ depend only on the present data and must be uniquely computable if the implication (5.4) is true. In conclusion, we shall represent w_* in Ω by the compact notation

$$w_* = \mathsf{A} \circ \Sigma + U\mathsf{B} \circ \mu^- + \mathsf{C} \circ \alpha, \tag{5.5}$$

where the domains of integration (under the $\circ$s) have been understood. This is an evolution equation for ψ in all respects, in the sense that it provides $\partial_t \psi$ at the present time in terms of the present data (ψ, f, T), according to

$$\partial_t \psi = \mathsf{A} \circ \Sigma + U(\mathsf{B} \circ \mu^- + 1) + \mathsf{C} \circ \alpha. \tag{5.5′}$$

A second step consists of formulating similar evolution equations for f and T, which are the only unknowns entering (5.5′) other than ψ. Of course, one could

† As it has been written down, (I6.7a*) contains two terms, on its left- and right-hand sides respectively, that can be seen to diverge for $\psi \to \psi_{\mathsf{a}}+$. This seemingly annoying fact does not raise difficulties, however, because it can be shown that these divergent terms exactly balance with each other in the limit $\psi \to \psi_{\mathsf{a}}+$. Hence (I6.7a*) can be 'renormalized' (so to say) just by subtracting a common divergent term from both sides. This results in a completely regular and equivalent equation.

‡ Examination of the circularly symmetric case throws considerable light on possible (sufficient) conditions that would ensure this assumption. Singling out such conditions in the general case is an interesting, precisely stated, technical problem upon which we shall not dwell here, but which is expected to have a precise, accessible answer. Anyhow, the non-degeneracy of concern is a very weak assumption – to the point that could be ignored in practice. Indeed, if (I4.11), together with its accessory conditions, has a unique (in the small) solution at $t = 0$, then by continuity, it will keep such a unique solution for a while after $t = 0$, say up to some $\tau > 0$. By construction, this is exactly equivalent to the non-degeneracy of the problem for w_* up to that τ.

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expect that new unknowns will come into play: but this does not in fact happen, and one eventually gets (through elimination of w_*, p and U in (I6.6a*) and (I6.2′*))

$$f_t = af_{\psi\psi} + bT_{\psi\psi} + \mathscr{F}, \tag{5.6}$$

$$T_t = cf_{\psi\psi} + dT_{\psi\psi} + \mathscr{T}. \tag{5.7}$$

On the right-hand sides of these equations (which hold for $\psi_{\mathbf{a}} < \psi < 0$), only the terms that depend on $(f_{\psi\psi}, T_{\psi\psi})$ as functions of them† have been written down explicitly, with (a, b, c, d) being given by

$$a \doteq \langle \eta_\parallel |\boldsymbol{\nabla}\psi|^2 \rangle, \tag{5.8a}$$

$$b \doteq 0, \tag{5.8b}$$

$$c \doteq f(\partial_T \rho)^{-1} \{k\langle \eta_\parallel - N\rangle - \tfrac{2}{3}\partial_p \rho \langle \mathscr{L}\rangle\}, \tag{5.8c}$$

$$d \doteq (\partial_T \rho)^{-1} \{k\langle \mathscr{L}\rangle + \tfrac{2}{3}\partial_p \rho \langle \mathscr{M}\rangle\}, \tag{5.8d}$$

and with $k \doteq \rho - \frac{5}{3}p\partial_p \rho$.

$\mathscr{F}$ and $\mathscr{T}$ represent all the remaining contributions to f_t and T_t respectively. We stress that, for the given $\psi = \psi(\boldsymbol{\xi})$, a, c and d are well-defined functions of f, T, f_ψ and T_ψ at ψ, whereas $\mathscr{F}$ and $\mathscr{T}$ also depend on f and T as well-defined (synchronous) functional transforms of them over $(\psi_{\mathbf{a}}, 0)$. Specifying these $\mathscr{F}$ and $\mathscr{T}$ explicitly is a little cumbersome, but straightforward.

Equations (5.6) and (5.7) must be complemented by the relevant, and available, accessory conditions, i.e. (3.2′) and (I6.10*). In what follows, we shall investigate whether (or possibly under which restrictions) (5.6), (5.7), (3.2′) and (I6.10*), together with the present ($\equiv$ at $t = 0$) data, give a well-posed evolution problem for f and T, of course thinking of $\psi = \psi(\boldsymbol{\xi})$ as given, as t increases from $t = 0$ on (at least until some sufficiently small $\tau > 0$).‡ First of all, examination of a, c and d readily shows the following.

(i) a, c and d all vanish at the magnetic axis ($\psi \to \psi_{\mathbf{a}}+$) like $|\boldsymbol{\nabla}\psi|^2$. This is obvious for a and d; for c, it is enough to note that

$$\eta_\parallel - \mathscr{N} = (\eta_\parallel - \eta_\perp)|\boldsymbol{\nabla}\psi|^2 |\mathbf{B}|^{-2},$$

and that the limit of f^2 at the axis is greater than zero by assumption (see LS90, p. 220, line 15).

(ii) Trivially, the two eigenvalues of

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$$

are a and d; but, whereas a is unconditionally greater than zero in $(\psi_{\mathbf{a}}, 0]$ because of $\partial_T \rho < 0$, $\partial_p \rho > 0$ (LS90, p. 222, line 21),§ d will be similarly greater than zero in the same $(\psi_{\mathbf{a}}, 0]$ if and only if

$$-\frac{3}{2}\frac{k}{\partial_p \rho}\langle \mathscr{L}\rangle - \langle \mathscr{M}\rangle > 0 \tag{5.9}$$

† As (5.6) and (5.7) show, these functions of $(f_{\psi\psi}, T_{\psi\psi})$ are linear.

‡ In any case, this evolution problem will be of the non-autonomous type, owing to the presence of the arbitrarily given control functions.

§ Here one also finds that $\rho(p \to 0+) = 0$. This is inconsistent with the requirement that $\rho^- > 0$. To avoid this trivial difficulty, it is sufficient to replace $p = 0$ (in Ω_v) by $p = p^0$ (a possibly small, positive constant), also modifying (I6.8) into (I6.8*); see point (c) of Sec. 4.

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(compare LS90, p. 231, unnumbered equation between (I6.2′) and (I6.10)). Reasonably, k should be negative, because it equals $-\frac{2}{3}\rho$ when ρ is proportional to p, as in an ideal plasma. In this case, we would get the simpler inequality

$$p\langle\mathscr{L}\rangle-\langle\mathscr{M}\rangle>0 \tag{5.9′}$$

in place of (5.9). We stress that the available thermodynamic inequalities (5.3) are of no help to ensure the condition (5.9), (or (5.9′)), so, if we want to ensure that this condition holds, we must assume it by itself, as was done in LS90. Of course, no problem would arise if $\mathscr{L}\geqslant 0$ (since $-\mathscr{M}>0$); but $\mathscr{L}<0$ cannot be excluded in general.† Reconsidering the inequality (5.3c) in this case $\mathscr{L}<0$, however, we can replace $\mathscr{L}$ by $-T\mathscr{L}$ to find

$$|\mathscr{L}|<\left(|\mathscr{M}|\frac{\eta_\perp}{T}\right)^{1/2}\left|\frac{\boldsymbol{\nabla}\psi}{f}\right|;$$

hence the condition (5.9′), say, would be certainly fulfilled if

$$\frac{p^2\eta_\perp}{T}\left|\frac{\boldsymbol{\nabla}\psi}{f}\right|^2\leqslant|\mathscr{M}|.$$

It is clear that $d>0$ would follow in any case from $\lambda_\perp>0$ being sufficiently large.

The reason that prompts us to require that $d>0$ is evident: under this condition, the system (5.6) (5.7) turns out to be 'parabolic-like', or 'abstract-parabolic', in the precise technical sense that the dependence of the terms on the right-hand sides as functions of the unknown second 'space' derivatives are linear, with the transformation matrix having positive eigenvalues. Even though we have not been able to find out precise and complete statements in this sense in the current mathematical literature, we consider it quite reasonable that, possibly under convenient technical restrictions, the accessory SCs that make an abstract-parabolic system like (5.6), (5.7) 'well conditioned'‡ are exactly the same as for a corresponding genuinely partial differential companion of it, i.e. for a generic quasilinear partial differential parabolic system with the same transformation matrix. The intuitive reason is that, in an abstract-parabolic equation or system the (synchronous) functional-transform dependence on the right-hand side or sides, being of integral nature over the space domain, does not need any additional information (as the second space derivatives do!) on the 'side' part of the boundary in order to be fully defined as t increases. To be more explicit, let us make reference to a single quasilinear abstract-parabolic equation, namely

$$u_t=au_{xx}+\mathscr{U}, \tag{5.10}$$

where $a=a(u,u_x,x,t)>0$ in the closed base interval I (say, for $x\in[-1,1]$) at $t=0$ (so for sufficiently small t too) and

$$\mathscr{U}=\mathscr{U}(u,u_x,x,t\,|\,u)\doteq\int_I dx'\,G(u,u_x,x,t;u',u'_x,u'_{xx},\ldots;x')$$

for some sufficiently smooth function G (here $u'\equiv u(x',t)$, etc.). Our conjecture

† According to classical transport theory, $\kappa_\wedge$ and $\eta_\wedge$ are always non-negative, but $\kappa_\perp$ can have either sign. Hence sign $\mathscr{L}$ is not universally fixed.

‡ That is, endowed with a unique, Hadamard-stable solution for sufficiently small t.

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is that (5.10) would give a well-posed evolution problem if associated with the same SCs on ∂I, for $t > 0$, which are appropriate to the companion partial differential quasilinear parabolic equation

$$u_t = a u_{xx} + \mathscr{U}^*, \tag{5.10$'$}$$

where $\mathscr{U}^*$ is any (sufficiently smooth) function of $(u, u_x, x. t)$.

We shall now examine the actual SCs available (i.e. (3.2′) and (I6.10*)) in this spirit, dealing with the system (5.6), (5.7) as if it were a genuine quasilinear partial differential parabolic system (having assumed the condition (5.9) to hold at $t = 0$, and hence for sufficiently small $t > 0$), and, of course, for $\psi = \psi(\boldsymbol{\xi}, t \geqslant 0)$ given under its 'intrinsic constraints'.†

To begin with, we recall that the system (5.6), (5.7) is singular at the left, moving extreme, $\psi = \psi_{\mathbf{a}}(t)$, so, as is well known from the general theory, we do not need any SCs at this extreme.‡ On the other hand, (3.2′) is quite an acceptable (quasilinear) SC of the third kind§ at the fixed regular extreme $\psi = 0$, since it can be solved trivially with respect to $T^-_{\psi}(\mathscr{M} < 0!)$. After the usual elimination of p, (I6.10*) becomes

$$\begin{aligned}\mathscr{P}|\Omega_V|^{-1} = (\mathscr{J}^0|\Omega_V|^{-1} + (\mathscr{J}^0)^{-1}\mathscr{J}^-_{\psi})[f^0\langle w_*\rangle^0 + a^0 f^-_{\psi} - \langle \eta_{\parallel}\rangle^- f^0 j^-] \\ + f^0\langle w_*\rangle^-_{\psi} + (a^-_{\psi} - U) f^-_{\psi} - (f\langle \eta_{\parallel}\rangle j)^-_{\psi} + a^0 f^-_{\psi\psi}\end{aligned} \tag{I6.10*$'$}$$

(here w_* and U have to be eliminated in the usual way). This equation results from an elaboration – intended to obtain a synchronous SC of some 'standard' kind – of the compatibility condition ensuring the existence of a (unique) solution s of (I6.4a,b), continuous across γ together with its normal derivative and fulfilling the boundary condition (I4.35),

$$\mathscr{P} = \oint d\Gamma\, \partial_N s = \int_{\Omega_P} d_2\xi\, (f_t + f_{\psi}\, \partial_t \psi) + f^-_t |\Omega_V|. \tag{5.11}$$

But evidently, the resulting SC (I6.10*′) does not belong to any of the standard kinds, since it essentially ($a^0 > 0!$) includes the second ψ-derivative $f_{\psi\psi}$ at $\psi = 0-$.‖ At an intermediate stage of its elaboration, however, the condition (5.11) reads

$$f^-_t = |\Omega_V|^{-1}[\mathscr{P} - \mathscr{J}^0(f^0\langle w^0_*\rangle + \langle K^{\parallel}\rangle^-)], \tag{I6.10*$''$}$$

and this gives the 'side' value of f_t, $f^-_t \equiv f^+_t \equiv f^0_t$, in terms of the present state, after w_* has been eliminated via (5.5). Though far from being standard, this

† We shall use this term for those (not necessarily synchronous) constraints that are expressed independently of the other unknowns (f, T); e.g. $\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\nabla^2\psi = 0$ in Ω_P and $\nabla^2\psi = 0$ in Ω_V, or $[\![\psi]\!] = 0$, $[\![\partial_n \psi]\!] = 0$, $[\![\partial_t \psi]\!] = 0$, the requirement of good shape, etc.

‡ The line $\psi = \psi_{\mathrm{a}}(t)$ will never be characteristic until $|d_t \psi_{\mathbf{a}}| < \infty$ (as we shall obviously assume). Also, note that for finite $d_t \mathbf{a}, d_t \psi_{\mathbf{a}}(t)$ reduces to $\partial_t \psi(\boldsymbol{\xi} = \mathbf{a}(t), t)$. The motion of $\mathbf{a}$ turns out to be governed by the ordinary differential equation given in LS90 at the beginning of Sec. 7. Alternatively, $\mathbf{a}$ can be determined just by solving $\boldsymbol{\nabla}\psi(\boldsymbol{\xi}, t) = 0$ for $\boldsymbol{\xi}, \forall t \geqslant 0$.

§ For simplicity making reference to the standard quasilinear parabolic equation (5.10′), we recall that the most familiar SCs to be given, say at the right extreme $x = 1$ (assumed regular) – and in practice the only ones mentioned in the current literature – consist of (i) giving $u^{1-} \equiv u(x \to 1-, t \geqslant 0)$ (this is usually said to be an SC of the first kind), (ii) giving the first derivative u^{1-}_x (an SC of the second kind) or (iii) giving a quasilinear relation between u^{1-} and u^{1-}_x, solvable with respect to u^{1-}_x (sometimes called an SC of the third kind).

‖ Also, the SC (I6.10*′) includes a (synchronous) functional-transform dependence on the unknowns via w_*. This point should not raise difficulties (on the basis of what was explained before), and will be ignored here.

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kind of SC can be shown to completely fit our purposes; the canonical case in this sense, referred to the basic heat equation in $1+1$ dimensions, $u_t = u_{xx}$, is illustrated in Cannon (1986): see ex. 7.4 on p. 76.†

In conclusion, for the given (intrinsically constrained) $\psi = \psi(\xi)$, we have T^-_ψ and $f^-_t \equiv f^0_t$ known as functions of (f^-, T^-, f^-_ψ), and functional transforms of (f, T) over $(\psi_{\mathbf{a}}, 0)$. This practically brings to a conclusion the discussion of the well-posedness of our evolution problem: if the (compatible) initial data (ψ, f, T) make our conjectures/assumptions hold, for reasons of continuity the latter will remain valid for a sufficiently small time $\tau > 0$, and so our system will evolve along a deterministic, Hadamard-stable trajectory until that τ. By *construction*, if it exists, the solution (ψ, f, T) $(0 < t \leqslant \tau)$ satisfies all synchronous constraints, in particular the quasilinear equation (I.4.11) for ψ.‡

Possibly, τ will mark the loss of the good shape of ψ. In this case, our computational procedure will not be allowed beyond τ, and we will be forced to stop. It is to be expected that this loss of good shape will coincide with the transition to a different, less simple topology of the (poloidal) magnetic field, for instance with the single elliptic point splitting into two similarly elliptic points and a hyperbolic point appearing in between. Furthermore, one could speculate that such a 'topological crisis' should be relatively smooth (no disruptions or similar violent phenomena) owing to the very nature of our model, which neglects all inertial effects. Expressed differently, what would really 'jump' at the topological crisis would not be the system evolution, but rather our ability to follow it by the relatively simple procedure we have illustrated.§ Finally, it must be stressed that the above conclusions have to be interpreted as very reasonable conjectures; even though they can (it is hoped) be proved under some convenient technical restrictions, we have no genuine theorems in this sense.‖

In concluding this section, we shall provide some simple comments about the determination of s and φ (after (ψ, f, T) have been determined as described above), the *gradients* of which are involved in the expression of the component of the electric field normal to z. Note that we are dealing with decoupled, synchronous problems, to be solved at every $t \geqslant 0$. As we have seen, s is a solution to (I6.4a) in Ω_P and (I6.4b) in Ω_V, under the interface conditions

† With reference to the usual quasilinear equation [system], one could call 'of the fourth kind' an SC giving the [a] first t-derivative on the side part of the boundary as a function of the unknown[s] and its [their] space first derivative[s]. Trivially, (I6.10*″) is exactly what one would get on elimination of $f^-_{\psi\psi}$ from (I6.10*′) by means of (5.6) itself, taken in the limit $\psi \to 0-$ (an evident vicious circle).

‡ In practice, by our procedure we have transformed a continuum of *quasilinear* problems for ψ (with t as parameter) into the union of an initial problem of the same type and a continuum of *linear* problems for $\partial_t \psi$, again with t as parameter. This possibility is quite standard in nonlinear analysis: the familiar asymptotic expansion procedures (with respect to a small parameter) to solve a nonlinear problem are an important example in this sense.

§ In principle, this procedure should still be feasible for more complex topologies, by matching the solutions in separate regions across the corresponding separatrices; but, of course, this possibility has just an academic interest.

‖ After all, it would not be too difficult to formulate a convenient 'essential' version of our global evolution problem, better suited to a mathematical investigation. The whole of this matter, as well as the more specific questions related to it – especially the inversion of (a convenient essential version of) the linear operator acting on w_* – should be of great interest for the applied mathematician.

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$[\![s]\!] = 0$ and $[\![\partial_n s]\!] = 0$ (see (I4.20′) and (I4.19b)) and the boundary condition $\partial_N s = -\beta$, (I4.35), so it is a solution of a standard, two-region, internal Neumann–Poisson problem, whose compatibility condition, (5.11), has already been obtained, being equivalent to (I6.10*″). Hence a unique solution exists up to an arbitrary summand depending on t at most, which does not influence $\boldsymbol{\nabla} s$.†

As far as $\varphi = \varphi(l, \psi)$ is concerned, everything should be clear after the discussion in Sec. 4, point (g). The condition $\langle Q \rangle = 0$, equivalent to (I6.1*), ensures the single-valueness of φ in $\Omega_P \backslash \{\mathbf{a}\}$, and so in the whole of $\bar{\Omega}$.

Summing up, for the given $k^{(\varphi)}(\psi, t)$ (with finite $k^{(\varphi)}(\psi_{\mathbf{a}} +, t)$), our evolution problem for the *physical* unknowns $(\mathbf{B}, \mathbf{E}, p, T)$ has 'practically'‡ been proved to be uniquely solvable and Hadamard-stable if $d(t = 0) > 0$ in $\Omega_P \backslash \{\mathbf{a}\}$, for sufficiently small t and for arbitrarily given control functions (α, β) $(\Gamma, t \geqslant 0)$ and given (constrained) initial data.

In terms of *physical* variables, the prescription of the initial (ψ, f, T) is equivalent to that of:

(i) $\mathbf{B} = \mathbf{B}(t = 0)$ (z-independent and $\neq 0$ in $\bar{\Omega}$), solenoidal inside some closed, simple curve γ everywhere tangent to $\mathbf{B}_P \doteq \mathbf{B} - \hat{\mathbf{z}} B_z$, harmonic between γ and Γ, continuous across γ, and with well-shaped poloidal flux;

(ii) $T = T(t = 0)$, z-independent and constant along the poloidal magnetic lines inside γ.

As expected, owing to the pre-Maxwell approximation, we find to be non-zero both the (volume) electric charge density in the plasma ($\nabla^2 \varphi$ turns out generally to be non-zero in Ω_P) and the (surface) charge density at the interface γ ($[\![\partial_n \varphi]\!]$ turns out generally to be non-zero too).

The fact that (α, β) $(\gamma, t \geqslant 0)$, and only they, are the control functions of our evolution problem,§ was suggested in LS90 (p. 221, lines 7–9). Obviously, the 'underdetermination in the plasma' mentioned there refers to the three arbitrary functions $(k^{(\varphi)}, k^{(v_\parallel)}, k^{(Q)})$ $(\psi, t \geqslant 0)$, which must be prescribed as well, aiming at the complete determination of the five (scalar) synchronous unknowns $(\varphi, \mathbf{v}, \mathring{q}_{T_\parallel})$, see Sec. 8 of LS90, and also Sec. 6 of the present paper.

6. Final comments, addenda and modifications

Up to now, we have reviewed Secs 1–6 of LS90, focusing ourselves on the well-posedness of the evolution problem. Some additional comments and marginal modifications now follow, concerning Secs 7–10 and the Appendices of LS90.

(a) Section 7 of LS90 was devoted to a tentative analysis of the linear problem for w, as it was formulated there. Although correct in itself, and apart from its practical worth, this analysis cannot be immediately adapted to the present new problem for w_*, owing to the interface condition (I4.25*) now

† The determination of s was shifted to that of χ in LS90, which is somewhat redundant with respect to our actual needs.

‡ As we have seen, both the non-degeneracy of the linear problem for w_* and the well-posedness of the abstract parabolic problem for (f, T) are very mild conjectures. From a practical point of view, the real problem is that of determining the resolvent kernels ($\mathsf{A}, \mathsf{B}, \mathsf{C}$), see point (a) in Sec. 6.

§ In principle, if there is some gas between γ and Γ, we have one more control function, namely the (uniform, but not necessarily time-independent) gas pressure.

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being substantially different from condition (I4.25′): clearly our previous approach has to be significantly modified. In principle, the possibility of similarly reducing the problem for w_* to the form of an equivalent Fredholm problem of the second kind can be shown to survive (most of Sec. 7 and the whole of Appendix C of LS90 are of use in justifying this statement); however, the proof is too laborious, and of questionable value for our purposes. So, here we prefer to leave open the problem of determining the resolvent kernels $(\mathsf{A}, \mathsf{B}, \mathsf{C})$, omitting any discussion of possible constructive methods in this sense.

(b) Section 8 of LS90 does not require any modifications, apart from the correction of a trivial misprint appearing in the expression for $\boldsymbol{\Xi}$, where $|\mathbf{B}|^{-1}$ should be changed to $|\mathbf{B}|^{-2}$. Also, here we should note that $\mathbf{B}\cdot\boldsymbol{\Xi} = 0$ and $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\Xi} = 0$, and so $\boldsymbol{\nabla}\cdot(k^{(\varphi)}_{\psi}\,\boldsymbol{\Xi}) = k^{\varphi}_{\psi}\,\boldsymbol{\nabla}\cdot\boldsymbol{\Xi}$. Furthermore, putting

$$\Theta \doteq k^{(v_{\parallel})} - k^{(\varphi)}_{\psi}\int_0 \frac{dl}{|\boldsymbol{\nabla}\psi|}\,\boldsymbol{\nabla}\cdot\boldsymbol{\Xi},$$

we have

$$\boldsymbol{\nabla}\cdot(\mathbf{B}\Theta) = \mathbf{B}\cdot\boldsymbol{\nabla}\Theta = -\,k^{(\varphi)}_{\psi}\,\boldsymbol{\nabla}\cdot\boldsymbol{\Xi},$$

and so $\boldsymbol{\nabla}\cdot(k^{(\varphi)}_{\psi}\,\boldsymbol{\Xi} + \mathbf{B}\Theta) = 0$: the arbitrary contribution to $\mathbf{v}$ is solenoidal. It can immediately be checked that the similar arbitrary contribution to $\mathbf{q}_{T_{\parallel}}$, $\mathbf{B}k^{(q)}$, is solenoidal as well.

Apart from these arbitrary contributions, the equations that determine the five (scalar) synchronous unknowns $(\varphi, \mathbf{v}_{\perp}, \mathring{v}_{\parallel}, \mathring{q}_{T_{\parallel}})$ are as follows:

$$\partial_t\varphi = \frac{Q}{|\boldsymbol{\nabla}\psi|} \tag{6.1}$$

for φ (with Q from (3.3)), equation (I4.13) for $\mathbf{v}_{\perp}$,

$$\rho^{-1}\rho_t + \rho^{-1}\rho_{\psi}(U-H) + \boldsymbol{\nabla}\cdot\mathbf{v}_{\perp} + \mathbf{B}\cdot\boldsymbol{\nabla}\frac{\mathring{v}_{\parallel}}{|\mathbf{B}|} = 0 \tag{6.2}$$

for $\mathring{v}_{\parallel}$, which – as is easy (but rather cumbersome) to prove – gives a single-valued $\mathring{v}_{\parallel}$ (taking into account (I6.2*), (I5.2) and (I4.15)), and

$$\tfrac{3}{2}(p_t + p_{\psi}\,\partial_t\psi) + \tfrac{5}{2}\boldsymbol{\nabla}\cdot(\mathbf{v}p) + \boldsymbol{\nabla}\cdot\mathbf{q}_{\perp} + \mathbf{B}\cdot\boldsymbol{\nabla}\left[\frac{\mathring{\mathbf{q}}_{T_{\parallel}} - \overbrace{(\boldsymbol{\nu}\cdot\mathbf{E}')}^{\circ}_{\parallel}}{|\mathbf{B}|}\right] = \mathbf{E}\cdot\mathbf{J} + S \tag{6.3}$$

for $\mathring{q}_{T_{\parallel}}$ (with $\mathbf{q}_{\perp} \doteq \mathbf{q} - \mathbf{q}_{\parallel}$ and $-\mathbf{q}_{\parallel} = (\boldsymbol{\nu}\cdot\mathbf{E}')_{\parallel} - \mathbf{q}_{T_{\parallel}}$). Again, by some cumbersome manipulations, one can check that (6.3) provides a single-valued $\mathring{q}_{T_{\parallel}}$ (taking into account (I6.3*), (I6.1*), (I4.7), (I4.3), (I4.11), (I5.2) and (I4.15)).†

(c) On restoring a (no longer vanishing) U in the stationary equations of Sec. 9 of LS90, one gets

$$\mathscr{A}\circ(\rho H) = U(\rho_{\psi} + \mathscr{J}^{-1}\,\mathscr{J}_{\psi}\,\rho) \tag{I9.1*}$$

in place of (I9.1),

† In the light of the above statements, (I6.2*) and (I6.3*) could have been *introduced* as (necessary and sufficient) conditions to ensure the (physically obvious) requirement of single-valuedness for $\mathring{v}_{\parallel}$ and $\mathring{q}_{T_{\parallel}}$ respectively, starting from the corresponding local balances (I2.5) and (I2.7). This alternative might be preferable from a logical point of view.

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$$S - Uj = -\mathscr{A} \circ (\tfrac{5}{2} pH - L) \tag{I9.2*}$$

in place of (I9.2), and

$$\langle K^{\parallel} \rangle = fU \tag{I9.3*}$$

in place of (I9.3). It can be proved that all the remaining equations, (I6.6a*), (I6.2′*), (I6.7a*) and (I6.10*), reduce to identities in the stationary state $\partial_t \equiv 0$ (taking into account (I9.1*)–(I9.3*)). In particular, one has to put $\mathscr{P} = 0$ in (I6.10*).

It should be remarked that (I9.3*) is valid in a gauge (s, φ) where $\boldsymbol{\nabla} s \equiv 0$ (or, rather, $s \equiv 0$ owing to the meaning of $s \equiv \partial_t \chi$). In other words, s has to be thought of as the (trivial) solution of $\nabla^2 s = 0$ in Ω, under the usual homogeneous interface conditions and the homogeneous boundary condition $\partial_N s = 0$. This is a convenient, but not mandatory, choice, because the $\mathbf{E}$ of a stationary pair $(\mathbf{E}, \mathbf{B})$ can quite be represented by (I4.7) with $\partial_t \psi = 0$ and generally non-zero $\boldsymbol{\nabla} s$, provided that s is harmonic. But, if the gauge with $\boldsymbol{\nabla} s = 0$ is used, one is no longer allowed to put $\varphi|_\Gamma \equiv 0$; the boundary condition (I4.32) must now be employed in the form $\partial_\Gamma \varphi = -\beta$, and the single-valuedness of φ in $\bar{\Omega}$ requires $\mathscr{P} = 0$, as mentioned before. In $\Omega_P \backslash \{\mathbf{a}\}$, φ has to be found through (6.1), with $Q = -fU + K^{\parallel}$, and for the given $k^{(\varphi)}(\psi)$; and in Ω_V, it has to be found by solving a Dirichlet problem with $\varphi|_\gamma^+ = \varphi|_\gamma^-$, and $\varphi(\Gamma) = -\int_0^\Gamma d\Gamma' \beta(\Gamma') + \varphi(0)$. With no loss of generality, the constant $\varphi(0)$ can be put equal to zero, since $k^{(\varphi)}(0-) \equiv \varphi(\psi = 0-, \gamma = 0)$ is already arbitrary. As for the boundary condition (I4.13), now reduced to $-U = \alpha$, it shows at once that $U (\equiv \text{const})$ is a datum, and that α cannot depend on Γ.

It remains to consider the boundary condition for ψ, the solution to (I4.11) in Ω_P (for $\psi < 0$, and for the given $j = j(\psi)$) and to (I4.9) in Ω_V (for $\psi > 0$), under the usual 'interface' condition $[\![\partial_n \psi]\!] = 0$ across the closed, simple curve $\psi = 0$. With the boundary values of $\psi = \psi(\Gamma)$ prescribed, this results into a standard, two-region quasilinear Dirichlet problem. Usually one prescribes B_N on Γ under the obvious constraint

$$\oint d\Gamma \, B_N(\Gamma) = 0,$$

and so

$$\psi(\Gamma) = -\int_0^\Gamma d\Gamma' \, B_N(\Gamma') + \psi(0).$$

If, in particular, Γ is thought of as bounding a perfectly conducting (in the $\boldsymbol{\xi}$ plane) external region then $B_N = 0$, and $\psi(\Gamma)$ reduces to a constant, to be given positive and large enough to guarantee that the curve $\psi = 0$ is inside Γ. This is a common choice when dealing with magnetohydrostatic plasma equilibria. In any case, the constant $\psi(0)$ must be given, satisfying the same requirements. A different approach, much more significant from the physical point of view, consists of giving the total plasma z-current, $\int_{\Omega_P} d_2 \xi j$, under the requirement that $\partial_\Gamma \psi|_\Gamma = 0$. This results in a somewhat non-standard quasilinear Dirichlet problem, which has been considered and solved in Berestycki and Brezis (1980).

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(d) The first paragraph of Sec. 10 of LS90 must be modified in an obvious way in the light of the above indications. The rest of the section contains general considerations about a quasilinear abstract evolution equation of the implicit type, which is of interest in both the previous (LS90) and present contexts.

(e) Appendix A of LS90 could now be ignored, since it deals with the interface conditions to be required for χ. Anyhow, the expression for ∇^2 to be found in the first sentence is incomplete, and should be corrected to

$$\nabla^2 = \partial_n^2 + \partial_\gamma^2 + C\partial_n + D\partial_\gamma,$$

where D is the curvature of the n-curve (normal to the family of the $\psi =$ const curves), which is positive if the curve is convex towards increasing γ. The above modification, however, does not affect the final results (I4.24a,b): in fact, it can be shown that one ends up with a linear evolution problem (along the 'normal' trajectories, with d_t^* time derivatives) for $[\![\chi]\!]$ and $[\![\partial_n \chi]\!]$, whose only solution corresponding to

$$[\![\chi]\!](t=0) = 0, \qquad [\![\partial_n \chi]\!](t=0) = 0$$

is the trivial one.

(f) As noted before, Appendix B of LS90 can be ignored following point (g) in Sec. 4 of the present paper.

(g) There is nothing to modify in Appendix C of LS90.

List of symbols (plus other information)

t time.

$\boldsymbol{\xi}$ a point of the sectional plane.

$|\mathbf{X}|$ modulus of vector.

$\hat{\mathbf{X}} \doteq \mathbf{X}/|\mathbf{X}|$.

$\Omega_P = \Omega_P(t)$ (open) simply connected section of the plasma column.

$\Omega_V = \Omega_V(t)$ (open) annular section of the section of the vacuum (or gas) region.

$\Omega \doteq \Omega_P \cup \Omega_V$.

Γ [γ] external, fixed boundary of Ω_V [moving boundary of Ω_P]; also, a length cordinate along it, oriented counterclockwise.

X_Γ [X_γ] component of a vector $\mathbf{X}$ along Γ [γ].

N [n] a length coordinate normal to Γ [γ], increasing towards the outside.

X_N [X_n] component of $\mathbf{X}$ along N [n].

$|^+$ [$|^-$] limit value to γ from outside [inside], for functions defined in Ω having unique continuous extension to γ.

$[\![\cdot]\!] \doteq |^+ - |^-$ jump across γ (with sign).

$|^0$ value at γ (for functions with zero jump across γ).

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$|_\Gamma$ limit value to Γ (for functions defined in Ω_V and having unique continuous extension to Γ).

$|_M$ component of a vector on the sectional plane, so that $\mathbf{X} = \mathbf{X}_M + \hat{\mathbf{z}} X_z$. If $\mathbf{X} = \mathbf{B}$ then $\mathbf{B}_M = \mathbf{B}_P$ (poloidal $\mathbf{B}$).

$\mathbf{B} = \mathbf{B}(\boldsymbol{\xi}, t), \mathbf{E} = \mathbf{E}(\boldsymbol{\xi}, t)$ magnetic field, electric field. Both of these are assumed to be $C^1(\Omega)$ and to have unique continuous extensions to γ (from both sides) and to Γ. According to standard notation, the latter property is written as $\in C(\bar{\Omega})$. In our model, $[\![\mathbf{B}]\!] = 0$ and $d\boldsymbol{\gamma} \times \mathbf{B}^0_M = 0$; moreover, $\boldsymbol{\nabla} \times \mathbf{B} \in (C^1(\Omega_P), C(\bar{\Omega}_P))$.

$\psi = \psi(\boldsymbol{\xi}, t)$ a special poloidal flux, $C^2(\Omega)$ and $C^1(\bar{\Omega})$, conventionally chosen (with no loss of generality) so as to vanish on $\gamma, \forall t \geqslant 0$. Then

$$\psi(\boldsymbol{\xi}, t) = \int_{\bar{\boldsymbol{\xi}}(t)}^{\boldsymbol{\xi}} d\boldsymbol{\xi}' \times \mathbf{B}_M(\boldsymbol{\xi}', t) \cdot \hat{\mathbf{z}},$$

where $\bar{\boldsymbol{\xi}}(t)$ is an arbitrary point on γ (the above integral is independent of the special curve joining $\bar{\boldsymbol{\xi}}(t)$ and $\boldsymbol{\xi}$, since $\mathbf{B}_M \times \hat{\mathbf{z}}$ is a gradient owing to $\boldsymbol{\nabla}_M \cdot \mathbf{B}_M = 0$).

$\mathbf{E}^{(s)}_M, \mathbf{E}^{(i)}_M$ a special Clebsch decomposition of $\mathbf{E}_M$, with $\boldsymbol{\nabla}_M \cdot \mathbf{E}^{(s)}_M = 0$, $\boldsymbol{\nabla} \times \mathbf{E}^{(i)}_M \cdot \hat{\mathbf{z}} = 0$, $\mathbf{E}_M = \mathbf{E}^{(s)}_M + \mathbf{E}^{(i)}_M$.

$-s = -s(\boldsymbol{\xi}, t)$ a special stream function of $\mathbf{E}^{(s)}_M = -\hat{\mathbf{z}} \times \boldsymbol{\nabla} s, C^2(\Omega)$ and $C^1(\bar{\Omega})$, with $[\![s]\!] = 0$ and $[\![\partial_n s]\!] = 0$.

$\varphi = \varphi(\boldsymbol{\xi}, t)$ a special potential of $\mathbf{E}^{(i)}_M = -\boldsymbol{\nabla}\varphi$, belonging to $C^1(\Omega_P - \{\mathbf{a}\})$, $C^2(\Omega_V)$, $C^1(\bar{\Omega}_V)$ and $C(\bar{\Omega}_P)$, with $[\![\varphi]\!] = 0$, but $[\![\partial_n \varphi]\!] \neq 0$ in general.

$U = U(t) \doteq \partial_t \psi(\boldsymbol{\xi}, t) - E_z(\boldsymbol{\xi}, t)$ (it can easily be seen that the right-hand side of this definition has vanishing gradient), so $\mathbf{E} = -\hat{\mathbf{z}} \times \boldsymbol{\nabla} s - \boldsymbol{\nabla}\varphi + \hat{\mathbf{z}}(\partial_t \psi - U)$.

$f \equiv B_z$ axial magnetic field.

$w \equiv \partial_t \psi$, belonging to $C^2(\Omega)$ and $C^1(\bar{\Omega})$, with $[\![w]\!] = 0$, but $[\![\partial_n w]\!] \neq 0$ in general.

$w_* \doteq w - U$.

ρ mass density, belonging to $C^1(\Omega_P)$ and $C(\bar{\Omega}_P), \rho > 0$ in $\bar{\Omega}_P$.

p pressure, belonging to $C^1(\Omega_P)$ and $C(\bar{\Omega}_P)$, with $[\![p]\!] = 0$.

$\mathbf{v}$ (vector) plasma velocity, belonging to $C^1(\Omega_P)$ and $C(\bar{\Omega}_P)$.

T plasma temperature, belonging to $C^2(\Omega_P)$ and $C^1(\bar{\Omega}_P)$.

$\mathbf{J} \doteq \boldsymbol{\nabla} \times \mathbf{B}$ (putting $\mu_0 = 1$) (vector) plasma current density, belonging to $C^1(\Omega_P)$ and $C(\bar{\Omega}_P)$.

$\mathbf{q}$ (vector) heat flux in plasma, belonging to $C^1(\Omega_P)$ and $C(\bar{\Omega}_P)$.

$(\mathbf{J}_T, \mathbf{q}_T)$ the part of $(\mathbf{J}, \mathbf{q})$ driven by $\boldsymbol{\nabla} T$.

$\mathring{X}_\parallel \doteq \hat{\mathbf{B}} \cdot \mathbf{X}$ for any vector $\mathbf{X}$.

$\mathbf{X}_\parallel \doteq \hat{\mathbf{B}} \mathring{X}_\parallel$ for any vector $\mathbf{X}$.

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$(\cdot)_{\parallel,\wedge,\perp}$ (with reference to a conductivity or resistivity tensor) parallel, 'Hall', normal (with respect to $\mathbf{B}$).

$\boldsymbol{\sigma}, \boldsymbol{\kappa}, \boldsymbol{\nu}, \boldsymbol{\lambda}$ electrical, electrothermal, thermoelectric, thermal conductivity tensors.

$\boldsymbol{\eta} \doteq \boldsymbol{\sigma}^{-1}$ (electrical) resistivity tensor.

u 'normal' speed of γ, positive if oriented outwards; also, a generic unknown.

$\mathbf{a}$ position of the magnetic axis in the sectional plane.

l arclength coordinate along $\psi(\boldsymbol{\xi}) = \text{const}$, oriented counterclockwise.

$\mathscr{J} = \mathscr{J}(\psi) \doteq \oint dl/|\boldsymbol{\nabla}\psi|$ the integral being along $\psi\ (\xi) = \psi$.

$\langle\cdot\rangle \doteq \mathscr{J}^{-1} \oint dl(\cdot)/|\boldsymbol{\nabla}\psi|$ magnetic-average linear operator.

subscript (t,ψ) (t,ψ)-partial derivative of a function of $(\boldsymbol{\xi}, t)$ constant along $\psi = \text{const}$. For brevity, we have denoted such functions of (ψ, t) by the same symbol as the original.

$\mathscr{A} \circ (\cdot) \doteq \mathscr{J}^{-1} \langle \mathscr{J}(\cdot) \rangle_{\psi}$.

$\mathbf{K}^{\perp}, K^{\parallel}$ see (I4.13) and (I4.14); note that $\mathbf{K}^{\perp} = -\mathbf{E}' \times \mathbf{B}$ and $K^{\parallel} = -\mathbf{E}' \cdot \mathbf{B}$.

H see (I4.16).

j shorthand for $\nabla^2\psi$.

$h \doteq f^2 + \frac{5}{3}p$ (also shorthand for $|\boldsymbol{\nabla}\psi|$).

$k \doteq \rho - \frac{5}{3} p \partial_p p$.

$\alpha \doteq E_z|_{\Gamma}$.

$\beta \doteq E_{\Gamma}|_{\Gamma}$.

$\mathscr{P} \doteq -\oint_{\Gamma} d\Gamma\, \beta$.

$\Omega(\psi, t)$ the (open) domain $\psi(\boldsymbol{\xi}, t) < \psi$ of the sectional plane; hence $\Omega_P = \Omega_P(t) = \Omega(0, t)$ and $\Omega_V = \Omega_V(t) \doteq \Omega \backslash \Omega_P(t)$.

$|\cdot|$ (with reference to a measurable domain) meas(.).

a, b, c, d coefficients on the right-hand sides of the system (5.6), (5.7), defined by (5.8).

$(\mathscr{N}, \mathscr{L}, \mathscr{Z}, \mathscr{M})$ see Sec. 5; specifically,

$\mathscr{N} \doteq (\eta_{\parallel} f^2 + \eta_{\perp} |\boldsymbol{\nabla}\psi|^2) |\mathbf{B}|^{-2}$;

$\mathscr{L} \doteq \boldsymbol{\eta} \otimes \boldsymbol{\kappa} |\boldsymbol{\nabla}\psi|^2 |\mathbf{B}|^{-1}$;

$\mathscr{Z} \doteq \boldsymbol{\eta} \otimes \boldsymbol{\nu} |\boldsymbol{\nabla}\psi|^2 |\mathbf{B}|^{-1} (= -T\mathscr{L})$;

$\mathscr{M} \doteq (\boldsymbol{\eta} \otimes \boldsymbol{\kappa}\nu_{\wedge} + \boldsymbol{\eta} \odot \boldsymbol{\kappa}\nu_{\perp} - \lambda_{\perp}) |\boldsymbol{\nabla}\psi|^2$.

$\mathscr{N}^* \doteq (\eta_{\perp} f^2 + \eta_{\parallel} |\boldsymbol{\nabla}\psi|^2) |\mathbf{B}|^{-2}$.

$\mathscr{H} \doteq (\eta_{\parallel} - \eta_{\perp}) f |\boldsymbol{\nabla}\psi|^2 |\mathbf{B}|^{-2}$.

$(\mathscr{F}, \mathscr{J})$ defined in (5.6) and (5.7).

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$\boldsymbol{\alpha}\otimes\boldsymbol{\beta} \doteq \alpha_\wedge \beta_\perp - \alpha_\perp \beta_\wedge$.

$\boldsymbol{\alpha}\odot\boldsymbol{\beta} \doteq \alpha_\wedge \beta_\wedge + \alpha_\perp \beta_\perp$.

$\delta^{(c)}, \delta^{(m)}, \delta^{(e)}$ right-hand sides of (I6.1), (I6.2), (I6.3).

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