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The low frequency scalar diffraction by an elliptic disc

By B. D. SLEEMAN
Department of Mathematics, University of Dundee, Dundee

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Summary. The problem of scalar Dirichlet diffraction of a plane wave by an elliptic disc is discussed. A scheme is given whereby the low frequency expansion of the scattered field may be readily obtained. Series expansions are obtained for the far-field amplitude up to and including the second order in the wave number. The first two terms of the scattering cross-section are also derived.

1. *Introduction.* In a previous paper (Sleeman (7)) the problem of scattering by an ellipsoid was treated. In that paper expressions for the far-field amplitude and scattering cross-section were obtained when the ellipsoid was placed in the presence of a plane wave incident upon it along one of its principal axes. The problem of diffraction of a plane wave at 'grazing' incidence by an elliptic disc arose as a special case. In this paper we adapt the methods of the author's above-mentioned work to the case of the diffraction of an arbitrary plane wave by an elliptic disc, considered as a limiting form of the ellipsoid.

The method of solution follows closely that formulated by Kleinman (4) and it turns out that the Rayleigh Series obtained is in terms of Lamé polynomials which have been adequately tabulated by Arscott and Khabaza (1). In fact the first few terms of the far-field amplitude and scattering cross-section are of such a simple nature that numerical results may be readily obtained.

The plan of this paper is as follows. In section 2 some preliminary results are stated, including a representation theorem due to Kleinman, and which is the basis upon which the problem is resolved. This representation theorem is used in section 3 to obtain a scheme whereby the successive terms of the scattered field may be evaluated. In section 4 the first three terms of the scattered field are derived, while section 5 is devoted to obtaining expressions for the far-field amplitude and scattering cross-section.

Finally, we remark that the analysis of this paper becomes increasingly more involved as successive terms in the scattered field are evaluated. However, most of the calculations follow much the same lines as those contained in Sleeman (7) and so we shall spare the reader the details and simply refer to the author's paper cited. The notation for the Lamé functions which arise in this paper are due to Arscott (2) and we refer to this book for details or to the author's previous paper where all the relevant results needed are given as a preliminary.

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2. *Preliminaries.* Introduce ellipsoidal coordinates (α, β, γ) related to Cartesian coordinates (x, y, z) by

$$\left.\begin{aligned} x &= Lk^2 \operatorname{sn}\alpha \operatorname{sn}\beta \operatorname{sn}\gamma, && \alpha \text{ from } -2K \text{ to } 2K, \\ y &= -Lk^2k'^{-1} \operatorname{cn}\alpha \operatorname{cn}\beta \operatorname{cn}\gamma && \beta \text{ from } K \text{ to } K+2iK', \\ z &= iLk'^{-1} \operatorname{dn}\alpha \operatorname{dn}\alpha \operatorname{dn}\gamma, && \gamma \text{ from } iK' \text{ to } K+iK', \end{aligned}\right\} \tag{2·1}$$

where k is the modulus of the elliptic function $\operatorname{sn}(z, k)$, $k'^2 = 1-k^2$ and L is an arbitrary constant.

In Sleeman (7), the scatterer was taken to be the ellipsoid defined by $\gamma = \gamma_0$, which in terms of Cartesian coordinates has the form

$$\frac{x^2}{k^2L^2 \operatorname{sn}^2\gamma_0} - \frac{y^2}{k^2L^2 \operatorname{cn}^2\gamma_0} - \frac{z^2}{L^2 \operatorname{dn}^2\gamma_0} = 1. \tag{2·2}$$

If (2·2) is written in the well-known form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{2·3}$$

we find $$\operatorname{sn}^2\gamma_0 = \frac{a^2}{a^2-b^2}, \quad k^2 = \frac{a^2-b^2}{a^2-c^2}, \quad L^2 = a^2-c^2.$$

The elliptic disc considered in this paper is obtained as a limiting form of the ellipsoid $\gamma = \gamma_0$. Specifically the elliptic disc is defined by

$$\gamma_0 = K + iK'. \tag{2·4}$$

Equation (2·2) then has the form

$$\frac{x^2}{L^2} + \frac{y^2}{k'^2L^2} = 1, \quad (z = 0), \tag{2·5}$$

where $$L^2 = a^2, \quad k^2 = 1-b^2/a^2.$$

Denote by S the surface $\gamma = \gamma_0$ and let V be the volume exterior to S. In ellipsoidal coordinates the distance between the two points $P(\alpha, \beta, \gamma)$, $P_1(\alpha_1, \beta_1, \gamma_1)$ where

$$P, P_1 \in \overline{V} = V \cup S,$$

will be denoted by $R(P, P_1)$. We further adopt the convention that the normal to S at any point is inward (directed out of V).

Throughout this paper a function $f(P)$ will be defined to be regular at infinity, in the sense of Kellog (3), if

$$\lim_{\gamma \to ik'} |\operatorname{sn}\gamma f(P)| < \infty, \quad \lim_{\gamma \to ik'} \left| \frac{\operatorname{sn}^2\gamma}{\operatorname{cn}\gamma \operatorname{dn}\gamma} \frac{\partial f(P)}{\partial \gamma} \right| < \infty. \tag{2·6}$$

We now state a representation theorem due to Kleinman (4) in a form most suitable for the problem of this paper.

THEOREM. *Any function $\omega(P)$ defined for all $P \in \overline{V}$ which is twice differentiable and regular at infinity in the sense of* (2·6) *satisfies the integral equation*

$$\omega(P_1) = \int_V G_0(P, P_1) \nabla^2\omega(P) \, dV + \int_S \omega(P_s) \frac{\partial}{\partial n} G_0(P_s, P_1) \, dS, \tag{2·7}$$

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where dV is the volume element, ∇^2 is the Laplacian expressed in coordinates (α, β, γ). The suffices s are used to denote surface values on S for which dS is an element. Finally, $\partial/\partial n$ is the inward normal to S expressed in the coordinates $(\alpha_s, \beta_s, \gamma_0)$.

In equation (2·7), $G_0(P, P_1)$ is the potential Green's function of the first kind subject to Dirichlet boundary conditions and may be written Sleeman(7) as

$$G_0(P,P_1) = \frac{-2}{k^3 L}\sum_{m,n} A_{m,n} Ep_n^m(\alpha,\beta)\, E_n^m(\gamma_<)\, F_n^m(\gamma_>)$$
$$+\frac{2}{k^3 L}\sum_{m,n} A_{m,n} Ep_n^m(\alpha,\beta)\frac{E_n^m(\gamma_0)}{F_n^m(\gamma_0)} F_n^m(\gamma_1)\, F_n^m(\gamma) \quad (m = 0, 1 \ldots, n, n = 0, 1, \ldots,), \tag{2·8}$$

where $\left.\begin{matrix}\gamma_< = \gamma_1 \\ \gamma_> = \gamma\end{matrix}\right\}$ if $K + iK' < \gamma_1 < \gamma$, $\left.\begin{matrix}\gamma_< = \gamma \\ \gamma_> = \gamma_1\end{matrix}\right\}$ if $iK' > \gamma_1 > \gamma$, (2·9)

and
$$A_{m,n} = \frac{Ep_n^m(\alpha_1,\beta_1)(-1)^{\sigma+\tau}}{(2n+1)k^{2\tau}}. \tag{2·10}$$

In (2·8), (2·10) the symbols E_n^m, F_n^m denote any one of the eight Lamé polynomials and Lamé functions of the second kind respectively. Further in (2·10) $\sigma = 0$ unless $\operatorname{cn} z$ appears as a factor of $E_n^m(z)$, in which case $\sigma = 1$. Similarly $\tau = 0$ unless $\operatorname{dn} z$ is a factor of $E_n^m(z)$ in which case $\tau = 1$. When we wish to stress which type of Lamé function is involved in the $A_{m,n}$ we shall write as follows

$$A_{m,n}^{(u)} = \frac{uEp_{2n}^m(\alpha_1,\beta_1)}{(4n+1)}, \quad A_{m,n}^{(s)} = s\frac{Ep_{2n+1}^m(\alpha_1,\beta_1)}{(4n+3)}, \quad \text{etc.} \tag{2·11}$$

3. *The application of the representation theorem to the problem of diffraction by an elliptic disc.* Consider the time harmonic diffraction problem for the surface S with Dirichlet boundary conditions. This may be considered as determining the perturbation ϕ_s of an incident field ϕ_i due to S. Specifically for a given ϕ_i we seek a function ϕ such that

(i) $\phi = \phi_i + \phi_s \; (P \in \bar{V})$,

(ii) $(\nabla^2 + \chi^2)\phi_s = 0 \quad (P \in V)$,

(iii) $\phi_i + \phi_s = 0 \; (P \in S)$, (3·1)

(iv) ϕ_s satisfies the Sommerfeld radiation condition at infinity. The time factor $e^{-i\chi t}$ is assumed and has been suppressed. In this paper we take ϕ_i to be the plane wave

$$\phi_i = \exp\{-i\chi(lx + my + nz)\}, \tag{3·2}$$

where $[l, m, n]$ are the direction cosines of the wave direction.

It is our intention to represent the scattered field ϕ_s with the integral representation (2·7). As in the problem of scattering by an ellipsoid (Sleeman (7), (8)), we put

$$\omega(P) = \exp(-i\chi k L \operatorname{sn}\gamma)\,\phi_s. \tag{3·3}$$

On using (3·1) (ii) we find that

$$\nabla^2\omega(P) = -\chi^2\omega(P)\frac{[k^2\operatorname{sn}^2\gamma(\operatorname{sn}^2\alpha + \operatorname{sn}^2\beta) - k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta + 1 - (1+k^2)\operatorname{sn}^2\gamma]}{k^2(\operatorname{sn}^2\gamma - \operatorname{sn}^2\alpha)(\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma)}$$
$$+\frac{i\chi}{k^3 L}\frac{\left[\omega(P)\dfrac{d}{d\gamma}(\operatorname{cn}\gamma\operatorname{dn}\gamma) + 2\operatorname{cn}\gamma\operatorname{dn}\gamma\dfrac{\partial\omega(P)}{\partial\gamma}\right]}{(\operatorname{sn}^2\gamma - \operatorname{sn}^2\alpha)(\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma)}. \tag{34}$$

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Hence on substituting $\nabla^2\omega(P)$ into (2·7) we obtain

$$\omega(P_1) = -\frac{\chi^2}{k^2}\int_V G_0(P,P_1)\frac{[k^2\,\mathrm{sn}^2\gamma(\mathrm{sn}^2\alpha+\mathrm{sn}^2\beta) - k^2\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta+1-(1+k^2)\,\mathrm{sn}^2\gamma]}{(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)}\omega(P)\,dV$$
$$+\frac{i\chi}{k^3L}\int_V G_0(P,P_1)\frac{\left[\omega(P)\frac{d}{d\gamma}(\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma)+2\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\frac{\partial\omega(P)}{\partial\gamma}\right]}{(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)}dV$$
$$+\int_S \omega(P_s)\frac{\partial}{\partial n}G_0(P_s,P_1)\,dS, \qquad (3\cdot5)$$

where
$$dV = ik^6L^3(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)\,d\alpha\,d\beta\,d\gamma,$$
$$\frac{\partial}{\partial n} = \frac{1}{k^2L}(\mathrm{sn}^2\alpha-\mathrm{sn}^2\gamma_0)^{-\frac{1}{2}}(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma_0)^{-\frac{1}{2}}\frac{\partial}{\partial\gamma}\Big|_{\gamma_0=\kappa+i\kappa'},$$
$$ds = ik^4L^2(\mathrm{sn}^2\alpha-\mathrm{sn}^2\beta)(\mathrm{sn}^2\alpha-\mathrm{sn}^2\gamma_0)^{\frac{1}{2}}(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma_0)^{\frac{1}{2}}\,d\alpha\,d\beta,$$

and $\omega(P_s) = -\exp(-i\chi kL\,\mathrm{sn}\,\gamma_0)\,\phi_i(P_s)$ using (3·1) (iii), (3·2). The problem of obtaining a low frequency expansion of the scattered field follows a procedure due to Kleinman (4). Let

$$\omega(P) = \sum_{t=0}^{\infty}\omega_t(P)\,\chi^t \qquad (3\cdot6)$$

and
$$\exp(-i\chi kL\,\mathrm{sn}\,\gamma)\,\phi_i(P) = \sum_{t=0}^{\infty}A_t(P)\,\chi^t, \qquad (3\cdot7a)$$

where
$$A_t(P) = \frac{1}{t!}\frac{d^t}{d\chi^t}[\exp(-i\chi k\,\mathrm{sn}\,\gamma)\,\phi_i(P)]|_{\chi=0}$$
$$= \frac{(-iL)^t}{t!}(k\,\mathrm{sn}\,\gamma + lk^2\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta\,\mathrm{sn}\,\gamma - mk^2k'^{-1}\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta\,\mathrm{cn}\,\gamma$$
$$+ ik'^{-1}n\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta\,\mathrm{dn}\,\gamma)^t, \qquad (3\cdot7b)$$

on using (3·2) and (2·1).

Substituting the above series into (3·5) and equating powers of χ to zero we obtain the scheme.

$$\left.\begin{aligned}
\omega_0(P_1) &= -\int_S A_0\frac{\partial}{\partial n}G_0(P_s,P_1)\,dS,\\
\omega_1(P_1) &= \frac{i}{k^3L}\int_V G_0(P,P_1)\frac{\left[\omega_0(P)\frac{d}{d\gamma}(\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma)+2\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\frac{\partial\omega_0(P)}{\partial\gamma}\right]}{(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)}dV\\
&\qquad -\int_S A_1\frac{\partial}{\partial n}G_0(P_s,P_1)\,dS,\\
\omega_{t+2}(P_1) &= -\frac{1}{k^2}\int_V G_0(P,P_1)\frac{[k^2\,\mathrm{sn}^2\gamma(\mathrm{sn}^2\alpha+\mathrm{sn}^2\beta) - k^2\,\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta+1-(1+k^2)\,\mathrm{sn}^2\gamma]}{(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)}\omega_t(P)\,dV\\
&\quad +\frac{i}{k^3L}\int_V G_0(P,P_1)\frac{\left[\omega_{t+1}(P)\frac{d}{d\gamma}(\mathrm{cn}\gamma\,\mathrm{dn}\,\gamma)+2\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\frac{\partial\omega_{t+1}(P)}{d\gamma}\right]}{(\mathrm{sn}^2\gamma-\mathrm{sn}^2\alpha)(\mathrm{sn}^2\beta-\mathrm{sn}^2\gamma)}dV\\
&\quad -\int_S A_{t+2}\frac{\partial}{\partial n}G_0(P_s,P_1)\,dS \quad (t\geqslant 0).
\end{aligned}\right\} \qquad (3\cdot8a,b,c)$$

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The results in this section have been obtained by the author in Sleeman(7) and we refer to that paper for details.

4. *The low frequency expansion of the scattered field.* The results of this section follow along much the same lines as those already obtained for the ellipsoid. We shall therefore refrain from giving details and simply state the results. However, it is fitting to remark that most of the calculations, although tedious, are reasonably straightforward. The evaluation of many of the integrals that arise is achieved by use of the orthogonality properties of Lamé polynomials (Arscott(2)).

We now proceed to give the first three terms of the scattered field in terms of $\omega_i(P_1)$ using equations (3·8a,b,c). Thus

$$\omega_0(P_1) = -\frac{uF_0^0(\gamma_1)}{uF_0^0(K+iK')}, \tag{4·1}$$

$$\begin{aligned}\omega_1(P_1) = ikL\Big\{&\frac{uF_0^0(\gamma_1)\,\mathrm{sn}\,\gamma_1}{uF_0^0(K+iK')} + \frac{uE_0^0(K+iK')\,uF_0^0(\gamma_1)}{\{uE_0^0(0)\}^2\{uF_0^0(K+iK')\}^2}\\ &- \frac{uE_0^0(\gamma_1)}{\{uE_0^0(0)\}^2\,uF_0^0(K+iK')} + l\,\frac{sF_1^0(\gamma_1)\,sEp_1^0(\alpha_1,\beta_1)}{sF_1^0(K+iK')\,sEp_1^0(K,K)}\\ &+ im\,\frac{cF_1^0(\gamma_1)\,cEp_1^0(\alpha_1,\beta_1)}{cF_1^0(K+iK')\,cEp_1^0(0,0)}\Big\},\end{aligned} \tag{4·2}$$

$$\begin{aligned}\omega_2(P_1) = -\frac{k^3L^2}{uF_0^0(K+iK')}\Big\{&\frac{A_{0,1}^{(u)}\,uF_2^0(\gamma_1)}{uEp_2^0(0,0)}\,\frac{\lambda^2}{\bar\lambda-\lambda}\int_{k+ik'}^{\gamma_1} uF_0^0(\gamma)\,uE_2^0(\gamma)\,\mathrm{sn}^2\gamma\,d\gamma\\ &- \frac{A_{1,1}^{(u)}\,uF_2^1(\gamma_1)}{uEp_2^1(0,0)}\,\frac{\bar\lambda^2}{\bar\lambda-\lambda}\int_{k+ik'}^{\gamma_1} uF_0^0(\gamma)\,uE_2^1(\gamma)\,\mathrm{sn}^2\gamma\,d\gamma\\ &+ \frac{A_{0,1}^{(u)}\,uE_2^0(\gamma_1)}{uEp_2^0(0,0)}\,\frac{\lambda^2}{\bar\lambda-\lambda}\int_{\gamma_1}^{iK'} uF_0^0(\gamma)\,uF_2^0(\gamma)\,\mathrm{sn}^2\gamma\,d\gamma\\ &- \frac{A_{1,1}^{(u)}\,uE_2^1(\gamma_1)}{uEp_2^1(0,0)}\,\frac{\bar\lambda^2}{\lambda-\bar\lambda}\int_{\gamma_1}^{iK'} uF_0^0(\gamma)\,uF_2^1(\gamma)\,\mathrm{sn}^2\gamma\,d\gamma\\ &- A_{0,1}^{(u)}\,\frac{uE_2^0(K+iK')\,uF_2^0(\gamma_1)}{uF_2^0(K+iK')\,uEp_2^0(0,0)}\,\frac{\lambda^2}{\bar\lambda-\lambda}\int_{K+iK'}^{iK'} uF_0^0(\gamma)\,uF_2^0(\gamma)\,\mathrm{sn}^2\gamma\,d\gamma\\ &+ A_{1,1}^{(u)}\,\frac{uE_2^1(K+iK')\,uF_2^1(\gamma_1)}{uF_2^1(K+iK')\,uEp_2^1(0,0)}\,\frac{\bar\lambda^2}{\bar\lambda-\lambda}\int_{K+iK'}^{iK'} uF_0^0(\gamma)\,uF_2^1(\gamma)\,\mathrm{sn}^2\gamma\,d\gamma\Big\}\\ &+ k^3L^2\Big\{A_{0,1}^{(u)}\,\frac{uF_2^0(\gamma_1)}{uEp_2^0(0,0)}\,\frac{\lambda^2\bar\lambda}{\bar\lambda-\lambda}\int_{K+iK'}^{\gamma_1} uE_2^0(\gamma)\,uF_0^0(\gamma)\,d\gamma\\ &- A_{1,1}^{(u)}\,\frac{uF_2^1(\gamma_1)}{uEp_2^1(0,0)}\,\frac{\lambda\bar\lambda^2}{\bar\lambda-\lambda}\int_{K+iK'}^{\gamma_1} uE_2^1(\gamma)\,uF_0^0(\gamma)\,d\gamma\\ &+ A_{0,1}^{(u)}\,\frac{uE_2^0(\gamma_1)}{uEp_2^0(0,0)}\,\frac{\lambda^2\bar\lambda}{\bar\lambda-\lambda}\int_{\gamma_1}^{iK'} uF_2^0(\gamma)\,uF_0^0(\gamma)\,d\gamma\\ &- A_{1,1}^{(u)}\,\frac{uE_2^1(\gamma_1)}{uEp_2^1(0,0)}\,\frac{\lambda\bar\lambda^2}{\bar\lambda-\lambda}\int_{\gamma_1}^{iK'} uF_2^1(\gamma)\,uF_0^0(\gamma)\,d\gamma\end{aligned}$$

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$$
\begin{aligned}
&- A^{(u)}_{0,1} \frac{uE^0_2(K+iK')\,uF^0_2(\gamma_1)}{uF^0_2(K+iK')\,uEp^2_0(0,0)} \frac{\lambda^2\bar{\lambda}}{\bar{\lambda}-\lambda} \int_{K+iK'}^{iK'} uF^0_2(\gamma)\,uF^0_0(\gamma)\,d\gamma \\
&+ A^{(u)}_{1,1} \frac{uE^1_2(K+iK')\,uF^1_2(\gamma_1)}{uF^1_2(K+iK')\,uEp^1_2(0,0)} \frac{\lambda\bar{\lambda}^2}{\bar{\lambda}-\lambda} \int_{K+iK'}^{iK'} uF^1_2(\gamma)\,uF^0_0(\gamma)\,d\gamma \Big\} \\
&+ \frac{kL^2 A^{(u)}_{0,0}}{uEp^0_0(0,0)} \Big\{ \frac{2}{3} \frac{uF^0_0(\gamma_1)\,uF'^0_0(\gamma_1)\,uE^0_0(\gamma_1)}{uF^0_0(K+iK')} \operatorname{sn}^2\gamma_1 \\
&- \frac{2}{3k^2} \frac{uE^0_0(K+iK')\,uF'^0_0(K+iK')\,uF^0_0(\gamma_1)}{uF^0_0(k+iK')} \\
&+ \frac{uF^0_0(\gamma_1)\,uF'^0_0(\gamma_1)\operatorname{sn}\gamma_1}{\{uF^0_0(K+iK')\}^2} - \frac{uF'^0_0(\gamma_1)\operatorname{sn}\gamma_1}{uF^0_0(K+iK')} - \frac{k}{\{uE^0_0(0)\}^2\{uF^0_0(K+iK')\}^2} \\
&+ \frac{kuF^0_0(\gamma_1)}{\{uF^0_0(K+K')\}^3\{uE^0_0(0)\}^2} - \frac{1}{3k^2} \frac{uE^0_0(\gamma_1)\{uF'^0_0(\gamma_1)\}^2\{\gamma_1 - E(\gamma_1)\}}{uF^0_0(K+iK')} \\
&+ \tfrac{1}{3} uE^0_0(K+iK') \frac{\{uF'^0_0(\gamma_1)\}^2}{k^2\{uF^0_0(K+iK')\}^2} uF^0_0(\gamma_1)\{K+iK'-E(K+iK')\} \Big\} \\
&+ \frac{L^2}{2} \Big\{ \frac{uEp^0_0(\alpha_1,\beta_1)\,uF^0_0(\gamma_1)}{uEp^0_0(0,0)\,uF^0_0(K+iK')} [(1-k^2m^2) + k^2(l^2-m^2)\lambda\bar{\lambda} + k^2m^2(\lambda+\bar{\lambda})] \\
&+ \frac{uEp^0_2(\alpha_1,\beta_1)\,uF^0_2(\gamma_1)}{uEp^0_2(0,0)\,uF^0_2(K+iK')} \Big[k^2 \frac{(l^2-m^2)\lambda^2\bar{\lambda}}{\bar{\lambda}-\lambda} + \frac{k^2m^2\lambda^2}{\bar{\lambda}-\lambda} \Big] \\
&- \frac{uEp^1_2(\alpha_1,\beta_1)\,uF^1_2(\gamma_1)}{uEp^1_2(0,0)\,uF^1_2(K+iK')} \Big[k^2 \frac{(l^2-m^2)}{\bar{\lambda}-\lambda} \lambda\bar{\lambda}^2 + \frac{k^2m^2\bar{\lambda}^2}{\bar{\lambda}-\lambda} \Big] \Big\} \\
&+ k^2L^2l \frac{sEp^0_1(\alpha_1,\beta_1)\,sF^0_1(\gamma_1)\operatorname{sn}\gamma_1}{sEp^0_1(K,K)\,sF^0_1(K+K')} + ik^2L^2m \frac{cEp^0_1(\alpha_1,\beta_1)\,cF^0_1(\gamma_1)\operatorname{sn}\gamma_1}{cEp^0_1(0,0)\,cF^0_1(K+iK')} \\
&- iL^2lm \frac{k'^2}{k^2} \frac{scEp^0_2(\alpha_1,\beta_1)\,scF^0_2(\gamma_1)}{scEp^0_2(K+iK',K+iK')\,scF^0_2(K+iK')}. \qquad (4{\cdot}3)
\end{aligned}
$$

In equation (4·3) the coefficients $A^{(u)}$ are defined by (2·11) and

$$\lambda, \bar{\lambda} = \{1 + k^2 \pm \sqrt{(1 - k^2.k'^2)}\}/3k^2.$$

Also in (4·3) the expression $E(\gamma)$ is given by

$$E(\gamma) = \int_0^\gamma \operatorname{dn}^2\gamma\,d\gamma. \qquad (4{\cdot}4)$$

From the above expressions we can easily write down the expansion of ω to the second order in χ using equation (3·3).

5. *The far-field amplitude and scattering cross-section.* In this section we use the first three terms of the scattered field ϕ_s to investigate the far-field amplitude for the elliptic disc.

We have

$$\begin{aligned}
\phi_s &= \exp(i\chi kL\operatorname{sn}\gamma)\,\omega(P) \\
&= \frac{\exp(i\chi kL\operatorname{sn}\gamma)}{\chi kL\operatorname{sn}\gamma}\{\chi kL\operatorname{sn}\gamma\omega(P)\}. \qquad (5{\cdot}1)
\end{aligned}$$

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From (5·1) it is clear that the far-field amplitude, defined as the coefficient of $e^{i\chi R/\chi R}$ is given by

$$f_s = \lim_{\gamma \to iK'} \{\chi kL \operatorname{sn}\gamma\, \omega(P)\}. \tag{5·2}$$

On carrying out the limiting process, using (4·1), (4·2), (4·3) we obtain

$$\begin{aligned} f_s = \chi kL \Bigg[& -\frac{1}{uE_0^0(0)\, uF_0^0(K+iK')} + \frac{i\chi kL}{\{uE_0^0(0)\}^2 \{uF_0^0(K+iK')\}^2} \\ & + \frac{k^2L^2\chi^2}{uF_0^0(K+iK')} \Bigg\{ \frac{uEp_2^0(\alpha,\beta)}{uE_0^0(0)\, uEp_2^0(0,0)} \frac{\lambda^2}{6(\bar{\lambda}-\lambda)} - \frac{uEp_2^1(\alpha,\beta)}{uE_0^0(0)\, uEp_2^1(0,0)} \frac{\bar{\lambda}^2}{6(\bar{\lambda}-\lambda)} \\ & + \frac{uEp_0^0(\alpha,\beta)}{uEp_0^0(0,0)} \Bigg(\frac{1}{\{uF_0^0(K+iK')\}^2 \{uE_0^0(0)\}^3} - \frac{(1+k^2)}{18k^2 uE_0^0(0)} + \frac{K-E-iE'}{3k^3\{uE_0^0(0)\}^2 uF_0^0(K+iK')} \\ & + \frac{1}{2k^3K}[k^2(l^2-m^2)\lambda\bar{\lambda} + k^2m^2(\lambda+\bar{\lambda}) - k^2m^2 - \tfrac{1}{3}] \Bigg) \Bigg\} \\ & + k^2L^2 \frac{l\chi^2 sEp_1^0(\alpha,\beta)}{sEp_1^0(K,K)\, sF_1^0(K+iK')\, sE_1^0(K)} + \frac{k^2L^2m\chi^2\, cEp_1^0(\alpha,\beta)}{cEp_1^0(0,0)\, cF_1^0(K+iK')\, cE_1^0(0)} + O(\chi^3) \Bigg]. \end{aligned} \tag{5·3}$$

If we redefine the far-field amplitude, f, as the coefficient of $e^{i\chi R/R}$ in the scattered field expansion then on using the definition of the scattering cross-section, σ, given by Morse & Feshbach (6) we have

$$\sigma = \int_\Omega |f|^2 d\Omega, \tag{5·4}$$

where the integration is performed over all solid angles at infinity. In terms of ellipsoidal coordinates

$$\sigma = -\frac{i}{2}\frac{k^2}{\chi^2} \iint_{\mathscr{S}} |f_s|^2 (\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta)\, d\alpha\, d\beta, \tag{5·5}$$

where f_s is given by (5·3) and $\iint_{\mathscr{S}} d\alpha\, d\beta$ denotes integration over

$$\alpha = -2K \text{ to } 2K, \quad \beta = K - 2iK' \quad \text{to} \quad K + 2iK'.$$

Thus on using (5·5), (5·3) and simplifying we obtain

$$\begin{aligned} \sigma = \frac{4\pi L^2}{K^2} \Bigg[1 + \chi^2L^2 \Bigg\{ & \frac{1+k^2}{9} - \frac{2}{3}\frac{K-E}{k^2K} \\ & - \frac{1}{K^2} - \frac{1}{k^2K}[k^2(l^2-m^2)\lambda\bar{\lambda} + k^2m^2(\lambda+\bar{\lambda}-1) - \tfrac{1}{3}] \Bigg\} + O(\chi^4) \Bigg]. \end{aligned} \tag{5·6}$$

Conclusions. We remark that one would expect to be able to verify the truth of the above results to a certain extent by reducing the elliptic disc to a circular one. This is simply achieved by allowing k to tend to zero. However this case is degenerate and much of the above analysis breaks down. There are two main reasons for this. First it is easily seen that when $k = 0$,

$$\frac{K+iK'-E(K+iK')}{k^2} = \frac{K-E-iE'}{k^2},$$

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in equation (5·3) is undefined. The second is that in the above general analysis, integrals which vanish identically because of the orthogonality of the Lamé polynomials may not, in all cases, do so when $k = 0$. This phenomenon was demonstrated when a similar limiting process was applied to the ellipsoid (Sleeman (7)) and so we shall not give the details here. A further reason for the breakdown in the results when $k = 0$, and one which is probably the seat of the trouble, is that the coordinate system (2·1) also breaks down.

The main advantage of the above results lies in their particularly simple form which enables numerical values to be obtained quite readily. This may be carried out in two ways, either

(i) the expansions are written in full and numerical tables of Jacobian elliptic functions are used, or

(ii) the tables available, Arscott and Khabaza(1), for Lamé polynomials are employed.

Finally, we remark that similar analysis may be employed in order to solve the Neumann problem for the elliptic disc, cf. Kleinman(5).

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