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Partitioning Euclidean space

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0. *Introduction*

$\mathbf{R}$ denotes the set of real numbers. For any finite n, $\mathbf{R}^n$ denotes n-dimensional space with the usual (Hilbert space) metric, d.

A subset $X \subseteq \mathbf{R}^n$ has *distinct distances* iff for all $w, x, y, z \in X$, $d(w,x) \neq d(y,z)$ unless $\{w,x\} = \{y,z\}$. Note that X can fail to have distinct distances in two different ways. One is that there is an *isosceles triangle* with vertex v; that is, a triple $\{v,x,y\}$ of distinct points of X with $d(v,x) = d(v,y)$. The other is a quadruple $\{w,x,y,z\}$ of distinct points of X with $d(w,x) = d(y,z)$.

P. Erdös and S. Kakutani [2] showed that, assuming CH, there is a partition of $\mathbf{R}^1$ into countably many pieces such that each piece has distinct distances. R. O. Davies [1] showed that the same result holds for $\mathbf{R}^2$. In § 2 we shall show, answering a question of Erdös, that the result holds in fact for $\mathbf{R}^n$ for any finite n.

If CH is false, then [2] shows that the result is false for all values of n.

The ingredient that our proof requires over the 2-dimensional result in [1] is a more careful study of planar and spherical surfaces in $\mathbf{R}^n$. We discuss this material in § 1, and give the proof in § 2. In § 3, we mention some open questions.

1. *Surfaces*

We shall now review some well-known facts of Euclidean geometry, and remark how they will be used in our construction.

The partition will be constructed by transfinite recursion. The construction will require us to consider certain surfaces. For example, consider how we make sure that each piece fails to contain an isosceles triangle $\{x,y,v\}$ with vertex v. If we have already placed v and x in one piece, and $d(v,x) = r$, then points on the *sphere* $S(v,r) = \{y : d(v,y) = r\}$, other than x, cannot be placed in that piece. On the other hand, if we have already placed x and y in one piece, and $x \neq y$, then points on the $n-1$ dimensional plane $\mathrm{BIS}(x,y) = \{v : d(v,x) = d(v,y)\}$ are forbidden to be placed in

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the same piece. More generally, we may describe planes of various dimensions as follows. If $x_0, \ldots, x_m$ are points in $\mathbf{R}^n$, they determine a plane

$$\mathrm{PL}(x_0, \ldots, x_m) = \{a_0 x_0 + \cdots + a_m x_m : a_0, \ldots, a_m \in \mathbf{R} \ \& \ a_0 + \cdots + a_m = 1\}.$$

The dimension of this plane is m if $x_0, \ldots, x_m$ are in general position, and less than m otherwise. We are including in the notion of plane the singletons, $\mathrm{PL}(x) = \{x\}$, as well as $\mathbf{R}^n$, the plane determined by $n+1$ points in general position.

If F is a subfield of $\mathbf{R}$, a *plane of* F is a plane, $\mathrm{PL}(x_0, \ldots, x_m)$ where $x_0, \ldots, x_m \in F^n$; equivalently, these are the solution sets to sets of linear equations in n variables with coefficients in F. By linear algebra, the intersection of two planes of F is a plane of F, and any plane $\mathrm{BIS}(x, y)$ is a plane of F whenever $x, y \in F^n$ and $x \neq y$.

A *sphere of* F is a sphere $S(v, r)$, where $v \in F^n$ and $r \in F$. Call F *quadratically closed* iff $\sqrt{|a|} \in F$ whenever $a \in F$. Let $\mathscr{L}_F$ be the set of all surfaces which are either planes of F or *spherical surfaces* of the form $S(v, r) \cap H$, where H is a plane of F and $S(v, r)$ is a sphere of F. Observe that if F is quadratically closed, we can always choose to represent such spherical surfaces with the centre of the sphere *in* H, since $S(v, r) \cap H = S(\tilde{v}, \tilde{r}) \cap H$, where $\tilde{v} \in H$ is the perpendicular projection of v onto H, and $\tilde{r}^2 = r^2 - d(v, \tilde{v})^2$. We use $\mathscr{L}$ for $\mathscr{L}_{\mathbf{R}}$. Note that $\mathscr{L}$ contains $\varnothing$ and $\mathbf{R}^n$, as well as all 1 and 2 element sets. Any element of $\mathscr{L}$ with more than 2 elements has size $2^{\aleph_0}$, and in fact is either a plane or a sphere or is isometric to a sphere in a smaller dimensional space.

1·1 LEMMA. *If F is quadratically closed, then the intersection of two elements of $\mathscr{L}_F$ is in $\mathscr{L}_F$.*

Proof. It is enough to prove this for spheres. By elementary geometry, if $S(v_1, r_1) \cap S(v_2, r_2)$ contains more than one point, then the intersection is of the form $H \cap S(v_3, r_3)$, where H is a plane of dimension $n-1$ perpendicular to the line between v_1 and v_2. Quadratic closure implies that H is a plane of F and $S(v_3, r_3)$ is a sphere of F. ∎

Observe that if H and K are infinite members of $\mathscr{L}$ and H is a proper subset of K, then $\dim(H) < \dim(K)$. It follows that

1·2 LEMMA. *No chain (under $\subset$) of members of $\mathscr{L}$ can have infinitely many members.*

Proof. Any chain can have at most $n+3$ members – namely, a 0, 1, and 2 element set, plus a surface of dimension j for each j between 1 and n. ∎

1·3 LEMMA. *If F is a quadratically closed subfield of $\mathbf{R}$, then $\mathscr{L}_F$ is a lattice in which $\wedge$ is $\bigcap$. $\mathscr{L}_F$ is complete and a complete sublattice of $\mathscr{L}$; in fact, if $\mathscr{S} \subseteq \mathscr{L}$, then there is a finite $\mathscr{S}_0 \subseteq \mathscr{S}$ such that $\bigvee \mathscr{S} = \bigvee \mathscr{S}_0$ and $\bigwedge \mathscr{S} = \bigwedge \mathscr{S}_0$.*

Proof. That $\wedge = \bigcap$ follows from Lemma 1·1. The rest follows from Lemma 1·2. ∎

Note that if $\mathscr{S} \subseteq \mathscr{L}_F$, then $\bigwedge \mathscr{S} = \bigcap \mathscr{S}$, but $\bigvee \mathscr{S}$ is the least element of $\mathscr{L}_F$ containing $\bigcup \mathscr{S}$, or $\bigcap \{H \in \mathscr{L}_F : \bigcup \mathscr{S} \subseteq H\}$. If $X \subseteq \mathbf{R}^n$, let $\mathrm{SUR}(X)$ be the least element of $\mathscr{L}$ which contains all points of X. In terms of the lattice operations, this is $\bigvee \{\{x\} : x \in X\}$. A special case of Lemma 1·3 is:

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1·4 LEMMA. *If F is quadratically closed and $X \subseteq F^n$, then* $\mathrm{SUR}(X) \in \mathscr{L}_F$. ∎

The inductive construction of a partition will assume we have the partition defined on some F^n for some countable subfield F, and we shall attempt to extend the partition to points outside of F^n. It will thus be important to study how these points lie with respect to F^n. If F is quadratically closed and $x \in \mathbf{R}^n \backslash F^n$, let $\mathrm{hull}_F(x)$ be the least element of $\mathscr{L}_F$ containing the point x; there is such a least element by Lemma 1·3. Call x a *planar* point of F if $\mathrm{hull}_F(x)$ is a plane, and a *spherical* point of F if $\mathrm{hull}_F(x)$ is spherical. Equivalently, x is a spherical point of F iff $d(x, v) \in F$ for some $v \in F^n$.

Now, assume that x is a spherical point of F. Call any $v \in F^n$ a *vertex point* for x (with respect to F) iff $d(x, v) \in F$. In our inductive construction, if we have already partitioned F^n, then a vertex point, v, represents a danger, since there will be infinitely many points, $y \in F^n$ with $d(y, v) = d(x, v)$; if any of these points happen to be in the same piece of the partition as is v, then x will have to be placed in a different piece, since $\{v, x, y\}$ is an isosceles triangle with vertex v. In our construction, rather than keeping track of the various y, we shall simply play it safe and always put x in different piece than any vertex point. To this end, let us now look at how these vertex points are arranged. The lowest dimension in which there can be a spherical point, x, of F, is dimension 2. In dimension 2, such an x can have only one vertex point (assuming F is quadratically closed). In dimension 3, there are really two possible cases for x. In the first, $\mathrm{hull}_F(x) = S(v, r)$, and v is the unique vertex point for x. But in the second, $\mathrm{hull}_F(x)$ is a circle, $S(v, r) \cap H$, where H is a 2-dimensional plane and $v \in H$. Then all points in F^3 on the axis of this circle are also vertex points for x.

Although our notation is different from that of Davies [1], the key difference between his construction in dimension 2 and ours in higher dimensions is that we must handle the possibility that there can be infinitely many vertex points for x. This could cause a problem, since we have decided that x be placed in a piece of the partition different from all its vertex points. We shall be aided by:

1·5 LEMMA. *If x is a spherical point of F, where F is quadratically closed, then there is a plane, H, of F, such that $H \cap \mathrm{hull}_F(x) = \varnothing$ and all vertex points of x with respect to F lie on H.*

Proof. Say $\mathrm{hull}_F(x) = S(v, r) \cap K$, where K is a plane of F and $v \in K$. Let v' be another vertex point for x, with $d(x, v') = r' \in F$. Then $x \in S(v', r')$, so $S(v, r) \cap K \subseteq S(v', r')$, which implies $(v - v') \perp K$. Thus, we may take H to be the plane of dimension $n - \dim(K)$ which is perpendicular to K and meets K at the point v. ∎

Actually, in Lemma 1·5, *all* points of $H \cap F^n$ are vertex points of x, but we shall not need this fact. One further result which we shall need is:

1·6 LEMMA. *If F is quadratically closed, $s \geqslant 0$, $H, K_1, \ldots, K_s$ are in $\mathscr{L}_F$, H is infinite, and each $H \backslash K_i \neq \varnothing$, then $(H \backslash (K_1 \cup \cdots \cup K_s)) \cap F^n$ is infinite. If G is a proper subfield of F, then $(H \backslash (K_1 \cup \cdots \cup K_s)) \cap (F^n \backslash G^n)$ is infinite.*

Proof. Use the fact that each $\dim(H \cap K_i) < \dim(H)$. ∎

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2. *Construction of a partition*

We express our partition by a function. Thus, the official statement of the theorem is:

THEOREM. *Assume* CH, *and let n be finite. Then there is a function* $\Phi: \mathbf{R}^n \to \omega$ *such that for each* k, $\Phi^{-1}\{k\}$ *has distinct distances.*

Proof. By CH, we can write $\mathbf{R}$ as a union of an increasing chain of countable quadratically closed subfields, $\mathbf{R} = \bigcup\{F_\alpha : \alpha < \omega_1\}$, with $F_\gamma = \bigcup\{F_\alpha : \alpha < \gamma\}$ for each limit γ. Let us write F_α^n for $(F_\alpha)^n$. We shall attempt to define $\Phi | F_\alpha^n$ by recursion on α. Of course, the main condition we wish to hold at stage α is

DD_α: For each k, $\Phi^{-1}\{k\} \cap F_\alpha^n$ has distinct distances.

However, it is easy to construct examples of a Φ which satisfies DD_α and cannot be extended to some point $x \notin F_\alpha^n$ (use the fact that x can have infinitely many vertex points in F_α^n). Rather, we shall specify a number of other conditions which will be seen to be stronger. The first of these are:

C0: $\Phi | F_0^n$ is 1–1.
$C1_\alpha$: All $\Phi | (F_{\alpha+1}^n \setminus F_\alpha^n)$ are 1–1.
$C2_\alpha$: If $x \in F_{\alpha+1}^n \setminus F_\alpha^n$ is a spherical point of F_α, then for all vertex points, v, of x with respect to F_α, $\Phi(x) \neq \Phi(v)$.
$C3_\alpha$: If $H \in \mathscr{L}_{F_\alpha}$, then $\Phi``(H \cap F_\alpha^n) = \Phi``(H \cap F_{\alpha+1}^n)$.

These conditions are sufficient by:

2·1 LEMMA. *If* Φ *satisfies C0 and* $C1_\alpha$, $C2_\alpha$, $C3_\alpha$ *for all* α, *then all* $\Phi^{-1}\{k\}$ *have distinct distances.*

Proof. We prove by induction on α that DD_α holds. DD_0 follows from C0, and DD_α is trivially preserved at limits, so we may assume DD_α and prove $DD_{\alpha+1}$. We note that $C1_\alpha$ greatly restricts the possible ways that $DD_{\alpha+1}$ can fail.

Let us first consider quadruples of distinct points, $\{v, x, y, z\}$, with $d(v, x) = d(y, z)$, where Φ has value k on $\{v, x, y, z\}$. By DD_α and $C1_\alpha$, exactly one of $\{v, x, y, z\}$, say x, is in $F_{\alpha+1}^n \setminus F_\alpha^n$. Then $r = d(y, z) \in F_\alpha$, and $x \in S(v, r)$, so v is a vertex point of x with respect to F_α and $\Phi(x) = \Phi(v)$, contradicting C2.

Next, consider isosceles triangles, $\{v, x, y\}$, with $d(v, x) = d(v, y)$, where Φ has value k on $\{v, x, y\}$. There are really two cases here. One is that v and y are in F_α^n and x is not. Then $d(v, x) = d(v, y) \in F_\alpha$, contradicting C2 as before. The other is that x and y are in F_α^n but v is not. In this case, v lies on $\mathrm{BIS}(x, y)$, which is in $\mathscr{L}_{F_\alpha}$. By C3, there is a $v' \in \mathrm{BIS}(x, y) \cap F_\alpha^n$ such that $\Phi(v') = k$, and then $\{v', x, y\}$ contradicts DD_α. ∎

Unfortunately, C0–C3 alone are not sufficient as inductive hypotheses in the construction of Φ. For example, if $x \in F_{\alpha+1}^n \setminus F_\alpha^n$ is a spherical point of F_α, there may be no possible values at all for $\Phi(x)$ which satisfy C2. We shall thus add the following condition for Φ, and shall show presently that it is sufficient to enable the construction to continue.

$C4_\alpha$: For each $s \geqslant 0$, if H, $K_1, \ldots, K_s$ are in $\mathscr{L}_{F_\alpha}$, H is infinite, and each $H \setminus K_i \neq \varnothing$, then $\Phi``(H \cap F_\alpha^n) \setminus \Phi``((K_1 \cup \cdots \cup K_s) \cap F_\alpha^n)$ is infinite.

We now conclude the proof of the Theorem by showing how to construct Φ to satisfy C0–C4. Let $\Phi | F_0^n$ be any 1–1 function into ω; this clearly satisfies C0 and

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$C4_0$. Since C3 implies that C4 is preserved at limits, it is enough to explain what is to be done at successor steps. Thus, assume that we have $\Phi|F^n_\alpha$ and that C0 and $C4_\alpha$ hold. We explain how to obtain Φ on the points of $D = (F^n_{\alpha+1} \backslash F^n_\alpha)$ so that $C1_\alpha$, $C2_\alpha$, $C3_\alpha$, and $C4_{\alpha+1}$ hold.

For $x \in D$, let $H_x = \text{hull}_{F_\alpha}(x)$. If H_x is planar, let $A_x = \Phi``(H_x \cap F^n_\alpha)$; this is infinite by $C4_\alpha$ (with $s = 0$). If $H_x$ is spherical, we apply Lemma 1·5 and let $V_x$ be a plane of $F_\alpha$ disjoint from $H_x$ which contains all vertex points of $x$, and then let $A_x = \Phi``(H_x \cap F^n_\alpha) \backslash \Phi``(V_x \cap F^n_\alpha)$; this is infinite by $C4_\alpha$ (with $s = 1$). So, for each $x \in D$, there is an infinite set of *allowable points*, A_x. If we extend Φ to D in any way so that $\Phi|D$ is 1–1 and each $\Phi(x) \in A_x$, we shall satisfy $C1_\alpha$, $C2_\alpha$, $C3_\alpha$, and the $s = 0$ case of $C4_{\alpha+1}$. However, in order to satisfy the general case of $C4_{\alpha+1}$, we must specify more precisely how we get $\Phi|D$. Enumerate D as $\{x_r : r \in \omega\}$, and enumerate $\mathscr{L}_{F_\alpha}$ as $\{T_r : r \in \omega\}$. Let $\Phi(x_r)$ be some element of A_{x_r} which is not in $\{\Phi(x_j) : j < r\}$, and which is not any of the $\Phi``(T_j \cap F^n_\alpha)$ such that $j < r$ and $x_r \notin T_j$; by $C4_\alpha$, there are infinitely many elements to choose from. We now must verify $C4_{\alpha+1}$.

Let H and $K_1, \ldots, K_s$ be elements of $\mathscr{L}_{F_{\alpha+1}}$, with each $H \backslash K_i$ infinite. For each i, let $\tilde{K}_i = K_i \cap F^n_\alpha$, and $\hat{K}_i = \text{SUR}(\tilde{K}_i)$. Then each $\hat{K}_i \subseteq K_i$. The $\hat{K}_i$ are in $\mathscr{L}_{F_\alpha}$ by Lemma 1·4. Say $\hat{K}_i = T_{j_i}$, and let $j = \max(j_1, \ldots, j_n)$. By Lemma 1·6, there are infinitely many r such that $x_r \in (H \backslash (K_1 \cap \cdots \cap K_s))$. For each of these r, $\Phi(x_r)$ is in $\Phi``(H \cap F^n_{\alpha+1})$, and is not in any of the $\Phi``(K_i \cap D)$ (since $\Phi|D$ is 1–1); furthermore if $r > j$, then, since $x_r \notin T_{j_i}$, $\Phi(x_r) \notin \Phi``(T_{j_i} \cap F^n_\alpha) = \Phi``(K_i \cap F^n_\alpha)$. There are thus infinitely many r such that $\Phi(x_r)$ is in $\Phi``(H \cap F^n_{\alpha+1})$, but not in any of the $\Phi``(K_i \cap F^n_{\alpha+1})$. ∎

We remark that Φ as constructed also has the property that whenever H and K are disjoint members of $\mathscr{L}$, $\Phi``H \cap \Phi``K$ is finite.

3. *Open questions*

We consider what variations on these results are possible.

Other spaces. For $n \geqslant 2$, we do not know for what other n-dimensional Banach spaces these results hold. They are easily seen to be false for the l^1 and l^∞ metrics on $\mathbf{R}^n$, or more generally, for any Banach space whose unit sphere contains a line segment. We do not know about the l^p metrics for $1 < p < \infty$. The result is false for any infinite-dimensional l^p, by an unpublished result of Pósa.

Other set theories. Under $\neg$CH, the result is false even if $n = 1$ by [2]; their proof uses a Hamel base to produce, in one piece of the partition, an x, y, z, t with $d(x, y) = d(z, t)$. Erdös asks whether one can, in ZFC, partition $\mathbf{R}^n$ (in the Hilbert space metric) into countably many pieces such that no piece contains an isosceles triangle; using a Hamel basis, this is easily seen to be true if $n = 1$.

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