Some extensions of a geometric inequality generated from Ptolemy's inequality

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1. *Introduction*

For convenience, throughout this Article, we shall use the standard notations for any triangle ABC, where:

1) the lengths of the sides BC, CA, AB are denoted by a, b, c, respectively,
2) the area of the triangle is denoted by S,
3) the semiperimeter of the triangle is denoted by s,
4) the circumradius and inradius of the triangle are denoted by R and r, respectively,
5) the lengths of the medians corresponding to vertices A, B, C are denoted by m_a, m_b, m_c, respectively,
6) the lengths of the angle bisectors corresponding to vertices A, B, C are denoted by l_a, l_b, l_c, respectively,
7) the lengths of the altitudes corresponding to vertices A, B, C are denoted by h_a, h_b, h_c, respectively.

In [1] Nguyen Xuan Tho proposed a geometric inequality as follows:

$$\frac{ab}{m_a m_b} + \frac{bc}{m_b m_c} + \frac{ca}{m_c m_a} \geqslant 4 \tag{1}$$

for any triangle ABC. Inequality (1) was proved in [1] by employing Ptolemy's inequality. More interestingly, two nice geometric inequalities by Martin Lukarevski (in [2, 3]),

$$\frac{a}{l_a} + \frac{b}{l_b} + \frac{c}{l_a} \geqslant 2\sqrt{3}$$

and

$$\frac{a^2}{l_a^2} + \frac{b^2}{l_b^2} + \frac{c^2}{l_a^2} \geqslant 4$$

are consequences of (1) (see [1]).

The aim of this Article is twofold. Firstly, we aim to generalise inequality (1) to hold for interior points of the triangle. Secondly, we improve inequality (1). We introduce two theorems as follows:

Theorem 1: Let P be an interior point of a triangle ABC. Let d_a, d_b and d_c be the distances from P to the sides BC, CA and AB, respectively. Then we have the following inequality:

$$\frac{d_a^2}{h_a^2} \cdot \frac{bc}{PB \cdot PC} + \frac{d_b^2}{h_b^2} \cdot \frac{ca}{PC \cdot PA} + \frac{d_c^2}{h_c^2} \cdot \frac{ab}{PA \cdot PB} \geqslant 1.$$

The equality holds if, and only if, triangle ABC is acute-angled and P coincides with its orthocentre.

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Theorem 2: For any triangle ABC, we have the following inequality:

$$\frac{ab}{4c^2 + a^2 + b^2} + \frac{bc}{4a^2 + b^2 + c^2} + \frac{ca}{4b^2 + c^2 + a^2} \geqslant \frac{1}{2}. \tag{2}$$

The equality holds if, and only if, triangle ABC is equilateral.

We see that inequality (1) follows as a consequence of both theorems above. Indeed, in Theorem 1, we set P to coincide with the centroid of triangle ABC. With the observation that in this case

$$\frac{d_a^2}{h_a^2} = \frac{d_b^2}{h_b^2} = \frac{d_c^2}{h_c^2} = \frac{1}{9},$$

and furthermore

$$PA = \frac{2}{3}m_a, \qquad PB = \frac{2}{3}m_b, \qquad PC = \frac{2}{3}m_c,$$

we readily obtain the inequality

$$\frac{bc}{m_a m_c} + \frac{ca}{m_c m_a} + \frac{ab}{m_a m_b} \geqslant 4,$$

which is precisely inequality (1).

In Theorem 2, we further notice that $m_a m_b \leqslant \frac{1}{2}\left(m_a^2 + m_b^2\right) = \frac{1}{8}\left(4c^2 + a^2 + b^2\right)$. Similarly, $m_b m_c \leqslant \frac{1}{8}\left(4a^2 + b^2 + c^2\right)$ and $m_c m_a \leqslant \frac{1}{8}\left(4b^2 + c^2 + a^2\right)$. Therefore

$$\frac{ab}{m_a m_b} + \frac{bc}{m_b m_c} + \frac{ca}{m_c m_a} \geqslant 8\left(\frac{ab}{4c^2 + a^2 + b^2} + \frac{bc}{4a^2 + b^2 + c^2} + \frac{ca}{4b^2 + c^2 + a^2}\right) \geqslant 4.$$

Thus, inequality (2) indeed represents a strengthened version of inequality (1).

2. *Proofs*

In this section, we shall provide proofs for Theorems 1 and 2.

Proof of Theorem 1: Let DEF be the Ceva triangle of P with respect to triangle ABC.

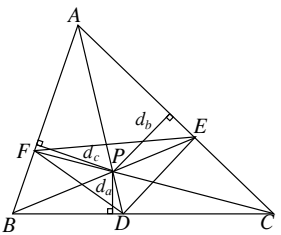


FIGURE 1: Proof of Theorem 1.

Applying the Ptolemy inequality to the quadrilaterals $AEPF$ and $BCEF$, respectively, we obtain the following inequalities:

$$PF \cdot AE + PE \cdot AF \geqslant AP \cdot EF, \tag{3}$$

and

$$BC \cdot EF + CE \cdot BF \geqslant BE \cdot CF. \tag{4}$$

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From (3) and (4), we deduce

$$BC \cdot \frac{PF \cdot AE + PE \cdot AF}{AP} + CE \cdot BF \geqslant BE \cdot CF. \tag{5}$$

Denote the areas of triangles PBC, PCA and PAB by S_a, S_b and S_c respectively. We have some length ratios in the triangle as follows:

$$\frac{AE}{AC} = \frac{S_c}{S_a + S_c}, \quad \frac{AF}{AB} = \frac{S_b}{S_a + S_b}, \quad \frac{CE}{CA} = \frac{S_a}{S_a + S_c}, \quad \frac{BF}{BA} = \frac{S_a}{S_a + S_b},$$

or equivalently,

$$AE = \frac{bS_c}{S_a + S_c}, \quad AF = \frac{cS_b}{S_a + S_b}, \quad CE = \frac{bS_a}{S_a + S_c}, \quad BF = \frac{cS_a}{S_a + S_b}, \tag{6}$$

and $\frac{PE}{PB} = \frac{S_b}{S_a + S_c}$, $\frac{BE}{PB} = \frac{S}{S_a + S_c}$, $\frac{PF}{PC} = \frac{S_c}{S_a + S_b}$, $\frac{CF}{PC} = \frac{S}{S_a + S_b}$,

or equivalently

$$PE = \frac{S_b}{S_a + S_c}PB, \quad BE = \frac{S}{S_a + S_c}PB, \quad PF = \frac{S_c}{S_a + S_b}PC, \quad CF = \frac{S}{S_a + S_b}PC. \tag{7}$$

Substituting (6) and (7) into inequality (5), we obtain

$$\frac{a}{PA} \cdot \left(\frac{S_c}{S_a + S_b}PC \cdot \frac{bS_c}{S_a + S_c} + \frac{S_b}{S_a + S_c}PB \cdot \frac{cS_b}{S_a + S_b} \right) + \frac{bS_a}{S_a + S_c} \cdot \frac{cS_a}{S_a + S_b} \geqslant \frac{S}{S_a + S_c}PB \cdot \frac{S}{S_a + S_b}PC.$$

This inequality is equivalent to

$$\frac{S_a^2}{S^2} \cdot \frac{bc}{PA \cdot PC} + \frac{S_b^2}{S^2} \cdot \frac{ca}{PC \cdot PA} + \frac{S_c^2}{S^2} \cdot \frac{ab}{PA \cdot PB} \geqslant 1. \tag{8}$$

With the observation that

$$\frac{S_a^2}{S^2} = \frac{d_a^2}{h_a^2}, \quad \frac{S_b^2}{S^2} = \frac{d_b^2}{h_b^2}, \quad \frac{S_c^2}{S^2} = \frac{d_c^2}{h_c^2},$$

from (8), we obtain the final inequality

$$\frac{d_a^2}{h_a^2} \cdot \frac{bc}{PB \cdot PC} + \frac{d_b^2}{h_b^2} \cdot \frac{ca}{PC \cdot PA} + \frac{d_c^2}{h_c^2} \cdot \frac{ab}{PA \cdot PB} \geqslant 1.$$

The equality holds if, and only if, both (3) and (4) are satisfied. This implies that P is the orthocentre of triangle ABC. If triangle ABC is obtuse, the orthocentre lies outside the triangle, and thus P must be an interior point of the triangle, which means the equality does not hold. Conversely, if triangle ABC is not obtuse, then equality holds when P coincides with the orthocentre of triangle ABC. This completes the proof of Theorem 1.

Proof of Theorem 2: We have:

$$\frac{ab}{4c^2 + a^2 + b^2} + \frac{bc}{4a^2 + b^2 + c^2} + \frac{ca}{4b^2 + c^2 + a^2} - \frac{1}{2} = \frac{\mathcal{A}}{2(4b^2 + c^2 + a^2)(4c^2 + c^2 + b^2)(4b^2 + c^2 + a^2)}$$

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where

$$\mathscr{A} = 2ab\left(4a^2+b^2+c^2\right)\left(4b^2+c^2+a^2\right)+2bc\left(4b^2+c^2+a^2\right)\left(4c^2+a^2+b^2\right)+$$
$$+2ca\left(4c^2+a^2+b^2\right)\left(4a^2+b^2+c^2\right)-\left(4a^2+b^2+c^2\right)\left(4b^2+c^2+a^2\right)\left(4c^2+a^2+b^2\right).$$

Write the expression $\mathscr{A}$ as

$$\mathscr{A} = \mathscr{B} + \mathscr{C}$$

where

$$\begin{aligned}\mathscr{B} = {} & 2(a+b-c)(a+c-b)(b+c-a)^2(b-c)^2 + \\ & 2(b+a-c)(b+c-a)(c+a-b)^2(c-a)^2 + \\ & 2(c+a-b)(c+b-a)(a+b-c)^2(a-b)^2\end{aligned}$$

and

$$\mathscr{C} = 5abc\left[\left(a+3b+3c-\frac{5bc}{a}\right)(b-c)^2+\left(b+3c+3a-\frac{5ca}{b}\right)(c-a)^2\right.$$
$$\left.+\left(c+3a+3b-\frac{5ab}{c}\right)(a-b)^2\right].$$

Obviously $\mathscr{B} \geqslant 0$, we need to prove $\mathscr{C} \geqslant 0$. Indeed, let

$$L_a = a+3b+3c-\frac{5bc}{a}$$

$$L_b = b+3c+3a-\frac{5ca}{b}$$

$$L_c = c+3a+3b-\frac{5ab}{c}.$$

We assume that $a \geqslant b \geqslant c$ then $ab \geqslant ac \geqslant bc$. We have

$$\begin{aligned}L_a &= a+3b+3c-\frac{5bc}{a} \\ &\geqslant 4b+3c-\frac{5bc}{a} \qquad (9) \\ &= \frac{4ab+3ac-5bc}{a} > 0 \ (\text{since } ab \geqslant ac \geqslant bc).\end{aligned}$$

If $2c \geqslant a$ then

$$\begin{aligned}L_b &= b+3c+3a-\frac{5ca}{b} \\ &\geqslant 4c+3a-\frac{5ca}{b} \\ &= \frac{4bc+3ab-5ca}{b} \\ &= \frac{4bc-2ab+5ab-5ac}{b} \geqslant 0 \ (\text{since } ab \geqslant ac \geqslant bc \text{ and } 2c \geqslant a).\end{aligned}$$

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If $2c < a$ then $2b > 2a - 2c > a$, we have

$$\begin{aligned} L_b &= b + 3c + 3a - \frac{5ca}{b} \\ &\geqslant 4c + 3a - \frac{5ca}{b} \\ &= \frac{4bc - 2ac + 3ab - 3ac}{b} > 0 (\text{since } ab \geqslant ac \geqslant bc \text{ and } 2b > a). \end{aligned}$$

Thus for all cases

$$L_b \geqslant 0. \tag{10}$$

We now consider

$$\begin{aligned} b^2L_b + c^2L_c &= b^2\left(b + 3c + 3a - \frac{5ca}{b}\right) + c^2\left(c + 3a + 3b - \frac{5ab}{c}\right) \\ &= b^3 + 3cb^2 + 3ab^2 - 5abc + c^3 + 3ac^2 + 3bc^2 - 5abc \\ &= (b + c)^3 + 3a(b^2 + c^2) - 10abc \qquad (11) \\ &\geqslant a(b + c)^2 + 6abc - 10abc \\ &\geqslant a(4bc) - 4abc = 0. \end{aligned}$$

Note that with $a \geqslant b \geqslant c$ and $b + c > a$ then we have

$$\frac{a - c}{a - b} \geqslant \frac{b}{c} \tag{12}$$

because this inequality is equivalent to

$$(b - c)(b + c - a) \geqslant 0.$$

From (9), (10), (11) and (12), we have

$$\begin{aligned} \mathscr{C} &= 5abc\left[L_a(b - c)^2 + L_b(c - a)^2 + L_c(a - b)^2\right] \\ &\geqslant 5abc(a - b)^2\left[L_b\frac{(c - a)^2}{(a - b)^2} + L_c\right] \\ &\geqslant 5abc(a - b)^2\left(L_b\frac{b^2}{c^2} + L_c\right) \\ &= 5abc(a - b)^2\frac{b^2L_b + c^2L_c}{b^2} \geqslant 0. \end{aligned}$$

We can easily see that the equality holds if, and only if, triangle ABC is equilateral. This completes the proof of Theorem 2.

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3. *Conclusion*

We can derive additional inequalities by considering specific points within the triangle when applying Theorem 1. Although the inequality in Theorem 2 is stronger than inequality (1), we can still strengthen it further. However, the inequalities we present below are merely conjectures; we have not yet obtained a complete proof for them.

For any triangle ABC, we have

Conjecture 1:

$$\frac{ab}{4c^2+a^2+b^2}+\frac{bc}{4a^2+b^2+c^2}+\frac{ca}{4b^2+c^2+a^2} \geqslant \frac{1}{2}+\frac{18S^2\left((b-c)^2+(c-a)^2+(a-b)^2\right)}{\left(4c^2+a^2+b^2\right)\left(4a^2+b^2+c^2\right)\left(4b^2+c^2+a^2\right)}.$$

Conjecture 2:

$$\frac{ab}{4c^2+a^2+b^2}+\frac{bc}{4a^2+b^2+c^2}+\frac{ca}{4b^2+c^2+a^2} \geqslant \frac{1}{2}+\frac{64r^2\left((b-c)^2+(c-a)^2+(a-b)^2\right)}{81s^2}.$$

Conjecture 3:

$$\frac{ab}{4c^2+a^2+b^2}+\frac{bc}{4a^2+b^2+c^2}+\frac{ca}{4b^2+c^2+a^2} \geqslant \frac{1}{2}+\frac{2r(R-2r)}{81R^2},$$

where R and s are the circumradius and inradius of triangle ABC respectively.

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